Modeling the Losses of Soil-applied Chemicals in Runoff: Lateral Irrigation versus Precipitation

RONY WALLACH, WILLIAM A. JURY,* AND WILLIAM F. SPENCER

ABSTRACT

The differences between rainfall- and surface irrigation-induced runoff are described in relation to the transfer of the surface-applied chemicals from soil solution to surface runoff water in the absence of infiltration. The different processes inducing runoff are characterized by the residence time distribution (RTD) of the water in the system. The time-dependent chemical concentration at the field outlet is affected by the RTD, even for the case of instantaneous equilibrium between chemicals in runoff water and soil solution. In this approach the water and chemical transport process in the field is represented by linear well-stirred mixing cells in series, for which a general solution for the effluent concentration is available. The rainfall- and the lateral irrigation-induced runoff processes, which are represented by complete mixing and plug flow models, respectively, form the two limiting cases for the actual chemical transport process by runoff. For each of these two extremes, mixing in the system is characterized by the parameter n, which is the number of cells in the series chosen to represent the actual transport process over the field.

Additional Index Words: Water quality, Chemical transport, Flood irrigation.

Losses of agricultural chemicals from soil to surface runoff water produce both a decrease in the effectiveness of the soil chemical treatment and a water quality hazard in the water which flows laterally off the field. Although there are numerous mathematical models which have been formulated to describe chemical release and transport in the runoff process, they only treat runoff which originates from spatially-uniform rainfall in excess of the infiltration rate of the soil (Bruce et al., 1975; Frere et al., 1975; Donigian et al., 1977; Knisel, 1980; Haith, 1980; Steenhuis and Walter, 1980). Furthermore, in these models the soil is represented as a thin surface mixing layer, and the runoff volume concentration is assumed to be proportional to the soil concentration at all times.

Parr et al. (1987) presented a diffusion-based model describing the release of chemical to runoff water. However, they did not model the runoff effluent concentration and made no comparisons with experimental data.

On agricultural fields, chemical losses in runoff may arise from furrow or border irrigation, in which water flows laterally across the entire field. The processes regulating the absorption of dissolved chemicals from soil water in this case are fundamentally different than during rainfall-induced runoff, where high quality water continually is striking the entire field area during the runoff event. Proper treatment of these two different surface conditions requires that the volume of runoff water be coupled with the soil water volume through a mass balance and a mechanistic representation of the chemical exchange process between the two volumes.

Many experiments have been conducted in order to define the influence of rainfall-runoff parameters on the release and transport of various chemicals from soil to runoff. These studies have investigated the effect of such variables as rain intensity and duration, infiltration rate, soil slope length, degree of slope, soil cover and chemical solubility and adsorption characteristics on the extent of chemical loss by runoff (Ingram and Woolhiser, 1980; Ahuja et al., 1981; Ahuja, 1982; Ahuja and Lehman, 1983; Heathman et al., 1985; Snyder and Woolhiser, 1985). In the models to date, however, which assume an instantaneous equilibrium between the chemical concentration in the soil solution of a thin mixing surface layer and the runoff water, the way in which these external parameters influence the concentration profile is often at odds with observation, particularly when runoff water flow rates are high enough to limit the time of exchange between the soil and overlying water.

Since the process of chemical loss in runoff during
lateral irrigation has not been modeled in previous investigations, it is important to establish similarities and differences between the previous approach used for the rainfall-induced process and the appropriate description of the irrigation-induced process. Specifically, it will be useful to characterize the relationship between the chemical concentration hydrograph and the water hydrograph, which may be fundamentally different for the same water discharge volume in the two processes.

This research effort will address those differences by developing a unified approach to describe the loss of chemicals in runoff under both types of water application. For simplicity in this presentation, the discussion will be confined to dissolved chemicals (e.g., nitrates) which do not adsorb to soil or sediments.

Because of the absence of experimental data and models for describing chemical runoff losses from furrow or border irrigated fields, the potential for and extent of chemical loss from soil in this situation as a first approximation must be estimated by comparison with the information available from rainfall-induced runoff studies. A natural way to compare these cases is to develop a process description of both systems which allows one to study the influence of the critical soil, environmental, and management factors on runoff volume and chemical effluent concentration for each type of runoff.

There are several basic differences in the characteristics of the two types of runoff process, and in the nature of the interaction with the soil zone underneath the moving water.

1. During a rainfall event, runoff starts when the intensity of the rainfall exceeds the infiltration rate. As a result, a certain amount of water will infiltrate prior to the onset of runoff. In contrast, in surface irrigation, water is supplied at the upstream end of the furrow or border at a sufficiently large lateral discharge rate that the maximum infiltration rate is exceeded from time zero onward.

2. During furrow or border irrigation, there will be less chemical mixing in the soil water near the soil surface than during rainfall, because in the former case there is no addition of kinetic energy by raindrop impact to the soil surface and soil water. This kinetic energy may greatly influence the extent of chemical mixing in the layer just below the soil surface.

3. In small watersheds, rainfall is applied simultaneously over the whole area, while during surface irrigation water is applied to the upper end of the field or furrow and advances downstream on a dry bed as a shock wave.

4. Rainfall is distributed randomly in time both between and during storms, whereas both the timing and duration of lateral irrigation are subject to management.

5. In a natural watershed the topography is non-uniform, with variable slopes and with both concave and convex surfaces which cause significant areal and temporal variability in infiltration water rates. The runoff resulting from such a heterogeneous regime may consist of both saturated and unsaturated subsurface lateral flow of water and dissolved chemicals which may even reappear on the surface. In surface irrigation, the slopes are small to moderate and relatively uniform.

This work will concentrate on the role of runoff water dynamics on the concentration profile at the field outlet, under the assumption of instantaneous equilibrium between the soil and runoff volumes. To facilitate an isolated examination of the influence of each of the relevant variables on the chemical transport process, infiltration will be neglected in the analysis.

**THEORY**

### Zero and Perfect Mixing Models

Although in principle it is necessary to have an exact knowledge of the water flow streamlines in both the soil and runoff volumes in order to predict chemical runoff losses from a land surface, in practice such detailed representation may not be necessary if only chemical outflow characteristics need to be described. An alternative to preparing such a detailed hydrologic description is to focus on the chemical output directly, and use transfer functions to represent the internal dynamics of complex systems.

**The Concept of Residence Time**

One may define the time period over which a unit volume of water is in contact with the soil surface as the *surface residence time*. There is an essential difference between the residence time over a field which receives spatially-uniform rainfall and over a field which receives irrigation water entering one side and leaving from the other. In the first case, water reaching the effluent end of the field will have a variable residence time which is a function of the distance traveled after it strikes the surface. Conversely, with surface irrigation, water reaching the effluent end of the field must travel across the entire field, and hence the average residence time of the water particles is approximately constant. In addition, the residence time distributions for both rainfall and lateral irrigation-induced runoff processes will depend on the degree of mixing within the runoff water (Himmelblau and Bischoff, 1980).

The residence time distribution RTD of runoff water above the soil surface is defined by a function \( f(t) \) such that \( \int_0^t dt \) is the fraction of the total water effluent volume which has a residence time between \( t \) and \( t + dt \). If runoff is induced by rainfall, the RTD may be estimated roughly for the case of zero mixing with the following simplifying assumptions: (i) Runoff begins everywhere at time zero while rain is falling at a uniform rate \( W \). (ii) The runoff height is constant and equal to the average water height over the field. (iii) Water flow is in steady state. Subject to these assumptions, the water velocity \( V(X) \) at any point \( X \) on the field between the upstream end (\( X = 0 \)) and the exit point (\( X = L \)) is

\[
V(X) = WX/H
\]

and the average residence time \( t(X) \) for an entry point \( X \) is

\[
t(X) = \int_X^L \frac{dX}{V(X)} = \frac{H}{W} \ln \left( \frac{L}{X} \right)
\]

or inversely

\[
X = \exp \left( - \frac{W}{H} t \right).
\]

The cumulative distribution function \( P(t) \) represents the probability that a runoff water element has a residence time less than or equal to \( t \). Since \( t \) is specified uniquely by the position \( X \) on the field, then
At steady state, \( P(t) \) is the fraction of the runoff water in the outflow that resides on the field surface less than time \( t \). Combining Eq. [3] and [4] gives

\[
\frac{dP(t)}{dt} = -\alpha P(t)
\]

where \( \alpha = W/H \).

Therefore, by definition, the residence time probability density function, \( f(t) \), is

\[
f(t) = \frac{dP}{dt} = \alpha \exp(-\alpha t)
\]

A plot of \( f(t) \) is shown in Fig. 1a for various values of \( \alpha \). The function \( f(t) \) can also be obtained directly as the outflow concentration declines exponentially with time during the event. Although the same RTD was obtained for both well mixed and plug flow representations of chemical transport, it should be noticed that both were formulated for the runoff water volume alone, and do not represent the soil–runoff system automatically. When the soil zone is assumed to be in equilibrium at all times with the chemical added in rainfall, the two water volumes (soil and runoff) degenerate into a single vertically well-mixed region and the above analysis applies, provided that the water height \( H \) is replaced by the total effective water thickness of the soil–runoff unit.

The assumption of a constant height of ponding \( H \) may perhaps be in error for horizontal surfaces, since water tends to build up under rainfall as the outlet end is approached. As an example of the opposite extreme, we may assume a constant water velocity in the runoff, so that the ponding height becomes

\[
H(X) = WX/V \quad 0 < X < L
\]

and the residence time, using Eq. [2], is simply

\[
t(X) = (L - X)/V.
\]

Using Eq. [4] and Eq. [6] we obtain in this case a cumulative residence time distribution

\[
P(t(X)) = \begin{cases} \frac{Vt}{L} & 0 < t < L/V \\ 1 & t > L/V \end{cases}
\]

or, letting \( H_{max} = WL/V \) and \( \alpha = W/H_{max} \), we obtain for the residence time probability density function (pdf) the function

\[
f(t) = \begin{cases} \alpha & 0 < t < 1/\alpha \\ 0 & t > 1/\alpha \end{cases}
\]

This pdf is a rectangle in contrast to the exponential pdf when \( H \) is assumed to be constant. Nevertheless, it is qualitatively similar, as shown in Fig. 1b.

The RTD for steady-state surface irrigation-induced runoff in the absence of chemical mixing is simply the \( \delta \)-function distribution

\[
f(t) = \delta(t - \tau)
\]

where \( \tau \) is the travel time required for the chemical added at the upstream end to reach the effluent boundary (Fig. 1c). If the water velocity \( V \) crossing the field is assumed to be uniform, then \( \tau = L/V \). At the other extreme, if instantaneous chemical mixing is assumed to occur in the water over the entire length of the field and if the velocity of the water crossing the field is assumed to be uniform and equal to \( V = Q/L \) where \( Q \) is the inlet volume flow per unit width, then the RTD function \( f(t) \) becomes that for a well-mixed reactor (Himmelblau and Bischoff, 1980)

\[
f(t) = (1/t_m) \exp(-t/t_m)
\]

where \( t_m = HL/Q = L/V \) is the mean residence time of the chemical. Thus, for surface irrigation, the residence time distributions for zero (Eq. [11]) and perfect mixing (Eq. [12]), in contrast to the rainfall case, have completely different shapes and properties. However, if \( Q \) in Eq. [12] is set equal to \( WL \) (so that the rainfall and irrigation processes add the same amount of water per unit time to the field), then Eq. [12] and Eq. [6] are identical. Thus, for a perfectly mixed system, a unit pulse of chemical added in rainfall has the same outflow shape as a unit pulse added in lateral irrigation.

In summary, earlier investigations modeling chemical movement in rainfall-induced runoff treated the runoff water over a watershed as a perfectly mixed reactor. For both the zero and perfectly mixed reactors the RTD is identical, however, so that this assumption is not critical when water enters the system over the field surface. It is inappropriate, however, to relate this type of distribution to the case where water is added to one side of the field and exits from another, as with surface irrigation, since this only describes the physically-unrealistic, perfectly mixed runoff volume model. In this case a more reasonable residence time distribution is the zero mixing delta function distribution (Eq. [11]).

Most of the runoff models which simulate the absorption of soil-applied chemicals by runoff water using a perfectly mixed reactor model, the soil and runoff volumes are assumed to be in instantaneous equilibrium. Thus, the chemical concentrations \( C_s \) of the soil water in a thin soil surface layer and in the runoff water \( C_r \) can be written as

\[
C_r = \beta C_s.
\]
The equilibrium constant $\beta$ is usually set equal to unity in most models (Donigian et al., 1977; Steenhuis and Walter, 1980). However, several studies assigned values of less than unity to $\beta$ to represent imperfect mixing between soil water and runoff water (Ingram and Woolhiser, 1980; Frere et al., 1980).

The two approaches illustrated thus far represent the extremes of zero and perfect mixing using the residence time distribution concept. For many practical applications, the degree of lateral mixing within the runoff water will not be well described by either approach. The next section will discuss two different methods, a convective-dispersive model and a linear mixing cell model, for representing the case of partial mixing.

**Partial Mixing Models**

**Convective-Dispersive Models**

In runoff flow systems, especially when the flow regime becomes turbulent, both convection and diffusion to some extent are important in chemical transport, and a partial dispersion model using convection and dispersion for non-reactive chemicals may be written as (Himmelblau and Bischoff, 1980)

$$\frac{(\partial c}{\partial t}) + V(\partial c}{\partial x}) - D(\partial^2 c}{\partial x^2}) = 0$$

where $c = c(x,t)$ is the local concentration at time $t$ and $D$ is the longitudinal dispersion coefficient which characterizes the degree of mixing within the fluid volume during flow. Equation [14] is the well-known convection-dispersion model often used in solute transport through porous columns (Biggar and Nielsen, 1967).

With $D = 0$, the convection-dispersion model Eq. [14] describes piston flow, which has the solution given in Eq. [11] for a unit pulse of tracer added at $x = 0$ at $t = 0$. For small, positive $D$, the model might represent highly turbulent flows in open tubes and channels. Such flows have flat velocity profiles and the axial dispersion coefficient $D$ can be identified as the eddy diffusivity. The dispersion coefficients for such models are usually estimated with tracer experiments by matching observations with the appropriate solution of Eq. [10]. When dispersion is important, an improved RTD over the ones described above may be obtained by solving Eq. [14] for a pulse input, with $D \neq 0$, which results in (Jury and Sposito, 1985)

$$f(t) = L / [2(\pi Dt)^{1/2}] \exp(-L - V\tau)^2 / 4Dt).$$

An alternative modeling approach which can be employed for studying dispersive flow systems is to combine well-mixed reactor cells in series. This model will be used below to compare the effect of dispersive mixing on the concentration outflow for rainfall and irrigation-induced runoff processes.

**Linear Cells in Series**

The linear cells in series model is well known from hydrology and chemical engineering. A linear cell is one which conforms to the principle of superposition with regard to its input and output, and the behavior of each element can be represented by its impulse response function. In hydrology the catchment is represented with this model by an assembly of cell units, each simulating a certain subcatchment by a lumped model unit. The representation of a watershed by a cascade of equal-sized reservoirs was first used by Nash (1957) as a basis for the conceptual interpretation of the unit hydrograph theory.

The following assumptions will be made in order to illustrate the different features between the different types of processes causing runoff:

1. There is instantaneous local equilibrium between chemicals in the soil water and chemicals in the surface runoff above the soil at a given point on the field (Eq. 13).
2. The dissolved chemicals are nonadsorbing and are initially present in a uniform concentration in the soil at time zero.
3. The field and overlying runoff water are represented as a channel consisting of a series of cells, each of which will be treated as a locally well-mixed reactor in equilibrium with the soil layer below it.
4. For the rainfall simulation, the rain intensity falling on each one of the reactor cells is the same.
5. For the rainfall simulation, each cell in the series receives both rainfall and runoff effluent from the adjacent upstream cell.
6. For the lateral irrigation simulation, the input to each cell is the output from the adjacent cell, which is assumed uniform.
7. No net infiltration of water into the soil occurs during either simulation.

The channel is divided into $n$ linear cells of the same shape and size which store equal amounts of water of volume $V_c$ which is partitioned into a runoff zone and a soil zone.

**Water Continuity**

Each cell $i$ in the series receives a runoff volume flow rate $R_{i-1}$ from the upstream cell, rainfall at a rate $W$ above, and releases runoff $R_i$ to the next cell in the series. If no storage changes occur in the cell, then

$$R_i = R_{i-1} + WA$$

where $A$ is the reactor surface area. If there is no inlet of water at the upstream end of the field, then Eq. [16] reduces to

$$R_i = iWA.$$  

For the case of lateral irrigation with no rainfall, the runoff volume is constant

$$R_i = Q$$

where $Q$ is the volume inflow rate.

**Chemical Continuity**

The general mass balance for the chemicals in the $i$th cell, assuming complete local mixing and that the chemical outflow from one cell is the inflow to the next one, can be written as

$$V_W(dC_i/dt) = R_{i-1}C_{i-1} - R_iC_i + WAC_p$$

where $C_i$ is the concentration of the chemical in the $i$th cell, $V_W$ is the water volume in the $i$th cell (constant for all cells), $R_i$ is the chemical concentration of the water entering with the rainfall. Combining Eq. [19] and [16] and assuming $C_p = 0$ produces

$$V_W(dC_i/dt) = R_{i-1}C_{i-1} - (W\cdot A + R_{i-1})C_i.$$  

Substituting Eq. [17] into [20] yields,

$$dC_i = (i-1)W\cdot C_{i-1} - i\cdot W\cdot C_i$$

where $H = V_W/A$ is the height of water in the cell. Writing the differential Eq. [21] for each cell in the series, produces a linear system of equations, called the state-space representation of a cascade of $n$ linear cells. Rewriting Eq. [21] in matrix notation produces
\[
\begin{bmatrix}
\frac{d}{dt} C_1(t) \\
\frac{d}{dt} C_2(t) \\
\vdots \\
\frac{d}{dt} C_d(t)
\end{bmatrix} =
\begin{bmatrix}
-K_1 0 0 \cdots 0 \\
K_1 - K_2 0 \cdots 0 \\
\vdots \\
0 \cdots K_{d-1} - K_d
\end{bmatrix}
\begin{bmatrix}
C_1(t) \\
C_2(t) \\
\vdots \\
C_d(t)
\end{bmatrix}
\]

where \( K_i = iW/H \).

The state-space representation for the lateral irrigation case is constructed in a similar manner. The continuity Eq. [19] for the \( i \)th cell for an upstream discharge feed at a uniform flow rate \( Q \) is obtained by setting \( R_i = Q, \ W = 0, \) or

\[ V_n (dC_i/dt) = Q(C_{i-1} - C_i) \]  

Rewriting the system of differential Eq. [19] in matrix notation gives, with \( Y = Q/V \n \)

\[
\begin{bmatrix}
\frac{d}{dt} C_1(t) \\
\frac{d}{dt} C_2(t) \\
\vdots \\
\frac{d}{dt} C_d(t)
\end{bmatrix} =
\begin{bmatrix}
-Y 0 \cdots 0 \\
Y - Y \cdots 0 \\
\vdots \\
0 \cdots Y - Y
\end{bmatrix}
\begin{bmatrix}
C_1(t) \\
C_2(t) \\
\vdots \\
C_d(t)
\end{bmatrix}
\]

where \( C_{in}(0) \) is the chemical concentration at \( x = 0 \) of the inlet solution.

The state equation for both cases, Eq. [22] and [24], may be written in symbolic vector notation as

\[
\frac{d}{dt} \mathbf{C}(t) = A \mathbf{C}(t) + \mathbf{b}(t) \]  

where \( A \) is an \( nxn \) square matrix, and

\( \mathbf{b}(t) = \) the input vector, which for the lateral irrigation case is simply a column vector with input into cell 1 only. For the rainfall case, where rainwater does not contain any dissolved chemicals, \( \mathbf{b} = 0 \).

The outflow from the system, equal to the downstream effluent discharge, is the outflow from the last cell, given by \( C_d(t) \) in Eq. [22] and [24].

Solution of the State Equations

For the constant flow and constant input case represented in this paper, the matrices described in Eq. [22] and [24] are time invariant. By direct analogy with the corresponding scalar differential equation a formal solution to Eq. [25] may be written as (Kailath, 1980).

\[
\mathbf{C}(t) = (\exp(A t) \mathbf{C}(t_0) + \int_{t_0}^{t} \exp(A(t - \tau)) \mathbf{b}(\tau) \, d\tau \]

where \( \mathbf{C}(t_0) \) is the initial state at \( t = t_0 \).

The exponential matrix \( \phi = \exp(At) \) may be regarded as the state transition matrix (STM) which, in the absence of any input, takes the initial state \( \mathbf{C}_0 \) to the state \( \mathbf{C}(t) \) by time \( t \). Some of the STM properties are described in Kailath (1980).

In general, computation of the STM is difficult, although there are well developed methods for solving it by numerical procedures which can be adapted from classical results in analysis, approximation theory and matrix theory. Moler and Van Loan (1978) proposed several methods for the computation of the exponential of a matrix.

For simplicity, addition of chemical with irrigation or rainwater will be neglected in the analysis to follow, and only the removal of chemical present in the soil and runoff water at \( t = 0 \) will be solved. Thus, the general solution becomes the first term on the right hand side of Eq. [26].

Although the general case involving a large number of cells is complicated, we may illustrate the principles adequately by restricting the number to three.

Solution for Uniform Rainfall

Equation [22] rewritten for three cells in series gives

\[
\frac{d}{dt} \begin{bmatrix}
C_1(t) \\
C_2(t) \\
C_3(t)
\end{bmatrix} =
\begin{bmatrix}
-K_1 0 0 \\
K_1 - K_2 0 \\
0 K_2 - K_3
\end{bmatrix}
\begin{bmatrix}
C_1(t) \\
C_2(t) \\
C_3(t)
\end{bmatrix}
\]

which has the general solution (see Appendix A)

\[
C(t) = a_1 e^{-K_1 t} \begin{bmatrix}
1/K_1 \\
K_1/(K_2 - K_1) \\
(K_2 - K_3)/(K_3 - K_1)
\end{bmatrix}
\]

\[
+ a_2 e^{-K_2 t} \begin{bmatrix}
0 \\
1 \\
K_2/(K_3 - K_2)
\end{bmatrix}
\]

\[
+ a_3 e^{-K_3 t} \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

where \( a_1, a_2, a_3 \) are scalar constants to be evaluated from the initial conditions.

In the rainfall case defined by the assumptions above, and for a spatially-uniform initial concentration

\[
\mathbf{C}(0) = C_0 \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

Eq. [28] reduces to the simple solution

\[
\mathbf{C}(t) = C_0 \exp(-Wt/H) \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

Equation [30] states that each cell of the reactor has the same concentration at all times, and hence that the solution in this case does not depend on the number of reactors in the simulation. This curious result is a consequence of the assumption that there is complete equilibrium between the
runoff water and the soil mixing zone within each cell. Since each cell receives the same rainfall input, has the same runoff flux entering and leaving the cell, and has the same initial concentration, all parts of the system have the same behavior as a function of time. In general, however, Eq. [26] will predict differences between cells if the initial distribution is nonuniform or if any lateral input of solute occurs. Significant differences between cells will arise if equilibrium between the soil and runoff zones is not assumed.

Solution for Lateral Feed

The upstream feed case (Eq. [24]), for a uniform initial concentration distribution (Eq. [29]) has the general solution (see Appendix A)

$$C(t) = C_0 \theta \exp(-\alpha t)$$  \[31\]

where

$$\theta = \sum_{j=1}^{n} (Yt)^j \frac{1}{K!}.$$  \[32\]

The resulting effluent concentration $C_n(t)$ is shown in Fig. 2 as a function of the number of cells $n$ in the simulation. When the number of the cells in the series is very large, the system behaves as a plug flow reactor.

**DISCUSSION**

The results from the mathematical solution show a marked difference in the concentration distribution with time for the two different runoff cases. It was noted for the case of a spatially uniform rainfall distribution (Eq. [29]) that the concentration output is independent of the chosen number of cells, and decays exponentially with time. The identical RTD obtained by both extreme models, where in one zero mixing is assumed while in the other perfect mixing is assumed, confirms that the soil-runoff volume under rainfall-induced runoff can be considered to be a perfectly mixed reactor from the water residence time point of view without introducing serious errors in outflow representation. In contrast, in the lateral irrigation case, the concentration output depends on the number of cells that the field is divided into. The number of cells is a measure of the extent of axial dispersion, which in the convection-dispersion model in Eq. [14] is represented by the Peclet number, $Pe = \frac{VL}{D}$ where $L$ is the field length.

The linear series model thus can be regarded as a numerical approximation of the axial dispersion model Eq. [14]. It has the advantage of permitting an arbitrary degree of complexity to be introduced into the state space representation without changing the form of the solution. For example, a spatially variable initial state $C_n$ will enter into the solution discussed above only at Eq. [29]. Equation [28] is still valid.

A dramatic contrast between the two types of runoff process may be illustrated for the case of an initial soil concentration even with the simplistic assumption of soil water-runoff water equilibrium. For the rainfall case, this results in an exponential decay of the outlet concentration from a maximum at $t = 0$. For the lateral irrigation case, water moving across the field will arrive at the initial soil concentration until the soil zone has been depleted enough to lower the effluent concentration. The soil concentration will be lowest at the upstream end of the field nearer to where the irrigation water enters the runoff stream (Fig. 3).

Although the calculations in this paper have only represented a simple case, the state space representation may be extended to cover more complex simulations, including nonequilibrium between the soil and runoff water, or for the case where infiltration is not neglected.

**APPENDIX A**

State Transition Matrix of the Cell Cascade

The general solution of the matrix Eq. [25] with $b = \theta$,

$$dC(t)/dt = A C$$  \[A1\]

will have the form (Himmelblau and Bischoff, 1980)

$$C(t) = \sum_{j=1}^{n} a_j \nu_j \exp(-\lambda_j t) \quad j = 1, 2, \ldots, n$$  \[A2\]

where $a_j$ are constants determined by the initial conditions and $\lambda_j$ are the eigenvalues satisfying

$$\det(A - \lambda I) = 0$$  \[A3\]

where $I$ is the unit matrix (1 along the diagonals and zero otherwise) and det is the determinant. The eigenvectors $\nu$ must satisfy the condition of orthogonality.
\[(A - \lambda_j I) \varphi_j = (0) \quad j = 1, 2, \ldots, n \quad [A4]\]

where \((0)\) is the null vector.

For the matrix \(A\) in Eq. [27], Eq. [A3] reduces to

\[-(K_1 + \lambda)(K_2 + \lambda)(K_3 + \lambda) = 0 \quad [A5]\]

or

\[\lambda_j = -K_j \quad j = 1, 2, 3 \quad [A6]\]

Thus,

\[
\begin{bmatrix}
0 & 0 & 0 \\
K_1 - (K_2 - K_1) & 0 & 0 \\
0 & K_2 - (K_3 - K_1) & 0
\end{bmatrix}
\varphi_1 = (0) \quad [A7]
\]

which is satisfied by the vector

\[
\varphi_1 = \begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix}
\]

Similarly,

\[
\varphi_2 = \begin{bmatrix}
0 \\
1 \\
(K_2 - K_1)
\end{bmatrix} [A9]
\]

and

\[\varphi_3 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix} [A10]\]

and

\[C(t) = a_1 \varphi_1 e^{-\lambda_1 t} + a_2 \varphi_2 e^{-\lambda_2 t} + a_3 \varphi_3 e^{-\lambda_3 t}. \quad [A11]\]

For the prescribed initial condition

\[C(0) = C_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [A12]\]

corresponding to uniform initial concentration and for \(K_j = jW/At\), \(j = 1, 2, 3\), the constants \(a_1, a_2, a_3\) have the values \(a_1 = C_{\theta}, a_2 = 0, a_3 = 0\), while \(\varphi_1\) has the value

\[\varphi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad [A13]\]

Thus, all parts of the field have the same concentration at a given time, which decays at a rate

\[C(t) = C_0 \varphi_1 \exp(-WAt/V). \quad [A14]\]

Thus, \(r = W/At\) is the field residence time for the chemical.

The system of equations for the lateral feed case, Eq. [24] with \(b = \varphi_0\), is

\[\frac{dC_i}{dt} + Y C_i = 0 \quad C_i(0) = C_0 \quad [A15]\]

\[\frac{dC_j}{dt} + Y C_j = YC_i \quad C_j(0) = C_0 \quad [A16]\]

\[\frac{dC_n}{dt} + Y C_n = YC_{n-1} \quad C_n(0) = C_0 \quad [A17]\]

which gives

\[y_j(t) = C_j e^{Yt} \quad [A18]\]

\[y_n(t) = C_0 \left( \sum_{k=0}^{n} \left( Yt \right)^K / K! \right) \quad [A19]\]

\[y_n(t) = C_0 \left( \sum_{k=0}^{n-1} \left( Yt \right)^K / K! \right) \quad [A20]\]
Knisel, W.G. (ed.). 1980. CREAMS; A field scale model for chemical, runoff, and erosion from agricultural management systems.