Derivation of equations which relate the effective surface charge density of a dielectric or electret to measurable parameters

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The measurement equations for the capacitive probe, dissectable capacitor, and vibrating capacitor methods of determining the effective or apparent surface charge density on a dielectric are derived as special cases of a single theory. The results agree with those of other authors.

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Measurement systems which determine the effective or apparent surface charge density on a dielectric from induced changes in the external electric field use equations derivable from the following theory. To simplify the boundary conditions without sacrificing much rigor, we have assumed that the electric fields are time invariant and independent of the y and z directions. References 1–4 discuss the extent to which this assumption holds in practice.

Suppose that a dielectric specimen exhibits the real charge densities shown in Fig. 1, and, in addition a total polarization

$$P''(x)=P(x)+P'(x),$$

where $P(x)$ is a slowly varying internal polarization and

$$P'(x)=\epsilon_0(\epsilon_2-1)E_2(x)$$

is an almost instanteously varying polarization. Then from the definition of electric displacement,

$$D_2(x)=\epsilon_0E_2(x)+P(x). \tag{1}$$

Integration of Poisson's equation,

$$\mathrm{div}[D_2(x)]=\rho(x),$$

gives another expression for the displacement:

$$D_2(x)=D_2(0)+\int_0^x\rho(x')\,dx'. \tag{2}$$

Specifically,

$$D_2(x_2)=D_2(0)+\int_0^{x_2}\rho(x')\,dx'. \tag{3}$$

Gauss's law, applied to the $x=0$ and $x=x_2$ faces of the specimen, gives

$$-\epsilon_0\epsilon_1E_1+D_2(0)=\sigma_1 \tag{4}$$

and

$$-D_2(x_2)+\epsilon_0\epsilon_3E_3=\sigma_2. \tag{5}$$

From Kirchoff's law,

$$E_1x_1+\int_0^{x_2}E_2(x)\,dx+E_3x_3-V=0, \tag{6}$$

where $V=V_p-V_g$.

A final boundary condition must be specified, relating the charge on the electrodes to the potential across them and to the field in the gap. Each measurement system has a different boundary condition. However, the types of measuring systems discussed in this paper have boundary conditions of the form

$$E_1=F-V/d, \tag{7}$$

where F and d are parameters, determined by the system geometry and measurement hardware, that split the field into voltage-independent and voltage-dependent components, respectively.

Substitution of Eqs. (1) and (2) into Eq. (6) gives an equation that, combined first with Eqs. (3)–(5) and then with Eq. (7) after the $\int_0^{x_2}\int_0^x\rho(x')\,dx'\,dx$ term is integrated by parts, yields

$$E_1=(1/\epsilon')\Big(\epsilon_0\epsilon_2\epsilon_3F+d^{-1}\Big\{-(Q/A')[\epsilon_2x_3+\epsilon_3(x_2-\chi)] + \epsilon_3\int_0^{x_2}P(x)\,dx-\sigma_1(\epsilon_3x_2+\epsilon_2x_3)-\sigma_2\epsilon_2x_3\Big\}\Big), \tag{8}$$

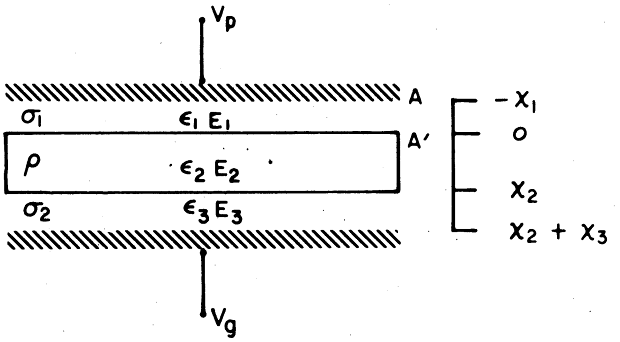

FIG. 1. Specimen-probe configuration for most systems that measure effective surface charge densities. The dielectric specimen exhibits real surface charge densities σ_1 and σ_2, volume charge density ρ (x), electric field E_2, face area A', and thickness x_2. The two gaps between the specimen and the electrodes are filled with dielectrics exhibiting relative permittivities ϵ_i and electric fields E_i, where $i=1$, 3. The metal electrodes at $x=-x_1$ and $x=x_2+x_3$ have area A and potentials V_p and V_g, respectively.

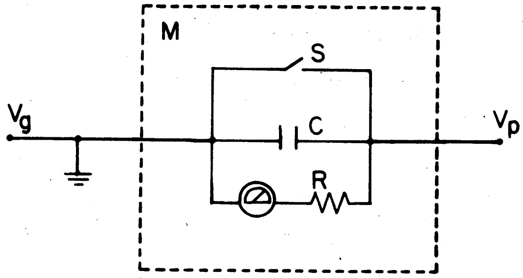

FIG. 2. Measurement hardware for capacitive probe and dissectable capacitor systems. S is a grounding switch, C and R are the internal capacitance and resistance of an electrometer M, and V_p and V_g refer to Fig. 1.

where Q is the total real volume charge,

$$\epsilon' = \epsilon_0[\epsilon_2\epsilon_3 + d^{-1}(\epsilon_2\epsilon_3 x_1 + \epsilon_1\epsilon_3 x_2 + \epsilon_1\epsilon_2 x_3)]$$

and

$$\chi = [\int_0^{x_2} x\rho(x)\,dx][\int_0^{x_2}\rho(x)\,dx]^{-1}.$$

Solutions for the other external parameters follow readily:

$$\int_0^{x_2} E_2(x)\,dx = (1/\epsilon')(\epsilon_0\epsilon_1\epsilon_3 x_2 F + \epsilon_3(Q/A')(x_2-\chi) + \sigma_1\epsilon_3 x_2 - \epsilon_3\int_0^{x_2}P(x)\,dx + d^{-1}\{(Q/A') \times [\epsilon_3 x_2 x_1 - \chi(\epsilon_3 x_1 + \epsilon_1 x_3)] - (\epsilon_3 x_1 + \epsilon_1 x_3) \times \int_0^{x_2}P(x)\,dx + \epsilon_3 x_2 x_1 \sigma_1 - \epsilon_1 x_2 x_3 \sigma_2\}), \quad (9)$$

$$E_3 = (1/\epsilon')(\epsilon_0\epsilon_1\epsilon_2 F + \epsilon_2(\sigma_1+\sigma_2+Q/A') + d^{-1}\{(Q/A')(\epsilon_2 x_1 + \epsilon_1\chi) + \epsilon_1\int_0^{x_2}P(x)\,dx + \epsilon_2 x_1\sigma_1 + \sigma_2[\epsilon_2 x_1 + \epsilon_1 x_2]\}), \quad (10)$$

$$V = (1/\epsilon')\{(Q/A')[\epsilon_2 x_3 + \epsilon_3(x_2-\chi)] - \epsilon_3\int_0^{x_2}P(x)\,dx + \sigma_1(\epsilon_3 x_2 + \epsilon_2 x_3) + \sigma_2\epsilon_2 x_3 + \epsilon_0 F(\epsilon_2\epsilon_3 x_1 + \epsilon_1\epsilon_3 x_2 + \epsilon_1\epsilon_2 x_3)\}. \quad (11)$$

Figures 1 and 2 show all but two components of a capacitive probe system.[4–6] Not shown is a grounded shutter located between the dielectric sample and the probe (the probe may be either of the two electrodes). Usually there is also a guard ring surrounding the sample. S is a grounding switch and C and R are values of the internal capacitance and resistance furnished by an electrometer M.

Opening the shutter, after first closing the switch S momentarily, draws a charge Q' from the capacitor onto the probe. Equation (11) and the probe potential

$$V = -Q'/C \quad (12)$$

determine the charge density of the sample. The values for F and $1/d$ in Eq. (11) follow from the substitution of

$$Q_1 = \epsilon_0\epsilon_1 A E_1 \quad (13)$$

into Eq. (12); i.e., the parameters in the charge-conservation boundary condition (7) for the capacitive probe system are $F=0$ and $1/d = C/(\epsilon_0\epsilon_1 A)$. Thus,

$$V = (1/\epsilon_0)\{(Q/A')[\epsilon_2 x_3 + \epsilon_3(x_2-\chi)] - \epsilon_3\int_0^{x_2}P(x)\,dx + \sigma_1(\epsilon_2 x_3 + \epsilon_3 x_2) + \sigma_2\epsilon_2 x_3\}[(C/A\epsilon_0) \times (\epsilon_2 x_3 + \epsilon_3 x_2 + \epsilon_2\epsilon_3 x_1/\epsilon_1) + \epsilon_2\epsilon_3]^{-1},$$

which agrees with the result of Wintle.[6] By setting $x_3 = 0$ and defining an effective surface charge density,

$$\sigma = \frac{Q}{A'}\left(1 - \frac{\chi}{x_2}\right) - \frac{1}{x_2}\int_0^{x_2}P(x)\,dx + \sigma_1, \quad (14)$$

one obtains Foord's result,[5]

$$V = \frac{x_2\epsilon_1\sigma A/C}{\epsilon_2 x_1 + \epsilon_1 x_2 + \epsilon_0\epsilon_1\epsilon_2 A/C},$$

which reduces to the result of Sessler and West[4] if the capacitance between the dielectric and the electrode is negligible.

Figures 1 and 2 show the components of a dissectable capacitor system.[1,7] Initially x_1 and x_3 are set to zero and the switch S is closed momentarily. Then the probe is moved to $x_1 \gg x_2$, thereby acquiring a charge density

$$Q'/A = -\sigma + CV/A. \quad (15)$$

If Eq. (13) is combined with Eq. (15) and the result is compared with Eq. (7),

$$\frac{1}{d} = \frac{C}{\epsilon_0\epsilon_1 A}, \quad F = -\frac{\sigma}{\epsilon_0\epsilon_1}.$$

For $\epsilon_1 = 1$, Eq. (11) becomes

$$V = -\frac{\sigma\epsilon_2 x_1/\epsilon_0}{(C/\epsilon_0 A)(x_2 + \epsilon_2 x_1) + \epsilon_2}.$$

By defining $\Delta V = [V]_{x_1 \gg x_2} - [V]_{x_1=0}$ one obtains the well-known relationship

$$\sigma = -C\Delta V/A.$$

Figures 1 and 3 show all the major components of a vibrating capacitor measurement system[3,8–10] except for a mechanical oscillator that drives the probe located at $-x_1$. When the electrode vibrates, a current runs through the external circuit unless the potentiometer supplies a voltage $V_B = -V_0$ to reduce the electric field E_1 to zero. V_0, the equivalent voltage, is the potential across the electrodes that produces an electric field equal to E_1. Dreyfus and Lewiner[8] give the relationship between E_1 and V as

$$\left(x_1 + \frac{\epsilon_1}{\epsilon_2}x_2 + \frac{\epsilon_1}{\epsilon_3}x_3\right)E_1 = V_0 + V.$$

Thus, $1/d$ and F in Eq. (7) become

$$\frac{1}{d} = -\left(x_1 + \frac{\epsilon_1}{\epsilon_2}x_2 + \frac{\epsilon_1}{\epsilon_3}x_3\right)^{-1}, \qquad F = V_0\left(x_1 + \frac{\epsilon_1}{\epsilon_2}x_2 + \frac{\epsilon_1}{\epsilon_3}x_3\right)^{-1}. \quad (16)$$

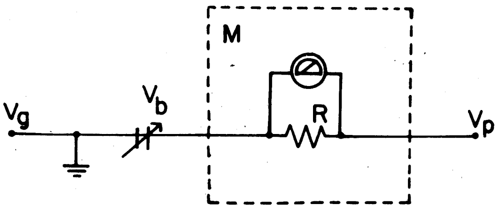

FIG. 3. Measurement hardware for vibrating capacitor system. R is the internal resistance of an electrometer M, V_B is an adjustable bias potential, and V_p and V_g refer to Fig. 1.

Substitution of Eq. (16) into Eq. (8) gives

$$V_0 = -\{(Q/A')[\epsilon_2 x_3 + \epsilon_3(x_2 - \chi)] - \epsilon_3 \int_0^{x_2} P(x)\,dx + \sigma_1(\epsilon_2 x_3 + \epsilon_3 x_2) + \sigma_2 \epsilon_2 x_3\}(\epsilon_0 \epsilon_2 \epsilon_3)^{-1},$$

which is the result of Dreyfus and Lewiner. If $x_3 = 0$, then, from Eq. (14),

$$V_0 = -\frac{x_2 \sigma}{\epsilon_0 \epsilon_2}$$

is the bias potential of Sessler[9]; $V_B = -V_0$ is the bias potential of Reedyk and Perlman.[10]

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