Numerical Solution of the Moisture Flow Equation for Infiltration into Layered Soils[1]

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ABSTRACT

A numerical solution of the moisture flow equation was devised and programmed for an IBM 650 computer. Solutions obtained for infiltration into a loam over a silt loam and vice versa showed that infiltration was governed by flow through the less permeable soil, provided the wetting front had extended well into the second layer. Solutions were also obtained for vertical upward and vertical downward infiltrations and for horizontal infiltration into two soils. The numerical solution was found to give excellent results when compared with the methods of E. J. Scott et al. and J. R. Philip for horizontal infiltration into homogeneous soils at a uniform initial water content.

THE MOISTURE FLOW EQUATION has been used by many workers to describe the isothermal flow of water into soil (4, 5, 6, 7). All of these methods require that the soil be homogeneous throughout. The method of Scott et al. (9) requires, in addition, that moisture diffusivity be exponentially or linearly related to moisture content. The method of Philip (7) is the only one treating vertical infiltration. This paper describes a method to estimate the solution of the moisture flow equation for vertical infiltration into layered soils without a specific mathematical relation between moisture diffusivity and water content.

THEORY OF THE METHOD

To meet the objective outlined above, it was necessary to devise a numerical solution of the flow equation. The numerical solution proved too tedious and time consuming for hand calculation. Consequently, the method was programmed on an IBM 650 computer.

The general form of the flow equation in one dimension can be written as follows:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial H}{\partial x}\right) \qquad [1]$$

Here, θ is water content, H is hydraulic head, x is distance, t is time, and K is conductivity. The numerical form of this equation for the vertical infiltration case discussed here is:

$$\frac{\theta_i^j - \theta_i^{j-1}}{\Delta t} = \frac{(h_{i-1}^{j-1} + h_{i-1}^{j} + 2G - h_i^{j-1} - h_i^j) K_{i-1/2}^{j-1/2}}{2(\Delta x)^2} - \frac{(h_i^{j-1} + h_i^j + 2G - h_{i+1}^{j-1} - h_{i+1}^j) K_{i+1/2}^{j-1/2}}{2(\Delta x)^2} \qquad [2]$$

where h is the pressure or tension head, K is the conductivity, the subscripts "i" refer to distance, the superscripts "j" refer to time, and G is the gravitational term. $G = \Delta x$ for vertical infiltration down, $G = -\Delta x$ for vertical infiltration up, and $G = 0$ for horizontal infiltration. Equation [2] is the Crank-Nicholson [3] equation adapted to include gravity.

An equation can be written for each depth increment involving unknowns of

$$\theta_i^j \text{ and } h_i^j, \ (i = 1, 2, \cdots, n-1)$$

provided estimates of K are made and the initial on boundary conditions are known. The initial conditions supply values of

$$\theta_i^{j-1} \text{ and } h_i^{j-1}, \ (i = 1, 2, \cdots, n-1)$$

The boundary conditions supply values of

$$h_o^j, \ h_n^j, \text{ and } \theta_n^j$$

A series of "n" equations can be formed having more than "n" unknowns. The equations cannot be solved specifically, so additional information is needed.

The derivation of equation [1] assumes a unique relation between the pressure or tension head, h, and moisture content θ. If this assumption holds, it is also possible to find a relation between θ and h, thus it is possible to write:

$$\frac{\theta_i^j - \theta_i^{j-1}}{\Delta t} \approx \frac{(h_i^j - h_i^{j-1}) C_i^{j-1/2}}{\Delta t} \qquad [3]$$

where the specific moisture capacity, C, is defined as

$$C_i^{j-1/2} = \left(\frac{\partial \theta}{\partial h}\right)_i^{j-1/2}$$

Substitution of equation [3] into [2] yields the final working equation:

$$\frac{h_i^j - h_i^{j-1}}{\Delta t} = \frac{(h_{i-1}^{j-1} + h_{i-1}^{j} + 2G - h_i^{j-1} - h_i^j) K_{i-1/2}^{j-1/2}}{2(\Delta x)^2 C_i^{j-1/2}} - \frac{(h_i^{j-1} + h_i^j + 2G - h_{i+1}^{j-1} - h_{i+1}^j) K_{i+1/2}^{j-1/2}}{2(\Delta x)^2 C_i^{j-1/2}} \qquad [4]$$

Provided reasonable estimates of C and K can be made, the series of "n" equations will have "n" unknowns and the solutions can be obtained. Arrays of such equations written in matrix form are tridiagonal. A very rapid method for solving tridiagonal matrices has been developed (8) and was used to complete the solution.

The critical part of the solution of equation [4] is the choice of K, C, and Δt. Once these estimates were available the solution was straight forward. The chosen values of K and C were held constant over each time interval, Δt, but were adjusted from one time interval to the next. A constant distance increment, Δx, was used throughout. For these computations Δx was 1 cm. or 2 cm. The number of distance increments used herein was arbitrarily taken as 20 ($n = 20$).

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The time increment, Δt, was variable. It is defined as the time required for a constant amount of water to enter the soil. Thus, at the start of an infiltration computation, Δt was small relative to values later in the infiltration run since the infiltration rate was high initially and decreased with time. The relationship is:

$$(\Delta t)^{j+1/2} = \frac{Q}{I^{j-1/2}} \quad [5]$$

where Q is a constant denoting the approximate amount of water entering the soil during each time increment and $I^{j-1/2}$ is the infiltration rate for the previous time increment. The value of $0.035\ \Delta x = Q$ was used herein. The conductivity was estimated as follows:

$$K^{j-1/2}_{i-1/2} = D^{j-1}_{i-1/2} \frac{(\theta^{j-1}_{i-1} - \theta^{j-1}_{i})}{(h^{j-1}_{i-1} - h^{j-1}_{i})} \quad [6]$$

where the diffusivity, D, is defined as:

$$D^{j-1}_{i-1/2} = \frac{\sum_{\theta=\theta_L}^{\theta^{j-1}_{i-1}} D\Delta\theta - \sum_{\theta=\theta_L}^{\theta^{j-1}_{i}} D\Delta\theta}{(\theta^{j-1}_{i-1} - \theta^{j-1}_{i})} \quad [7]$$

where θ_L is the lowest value of θ for which a D has been determined.

The use of equations [6] and [7] for estimating K is somewhat "round-about" in that an average diffusivity, D, must be first determined before K can be obtained. It would be more direct to estimate K from an equation similar to [7] in K instead of D. Equation [6] would not then be necessary. Preliminary tests of these two approaches showed that the use of equations [6] and [7] as constituted gave much better results. This is probably because D does not vary as widely as does K, which allows a better estimate of an average D than an average K.

For more convenient evaluation of equation [7], a table of θ vs. $\Sigma D\Delta\theta$ was constructed from the data of table 1 beginning with the lowest value of θ. For example the lowest value of θ for Sarpy loam for which data for D is given is 0.05. For $\theta = 0.05$, D is 1.88×10^{-4} and $D\Delta\theta$ would be $1.88 \times 10^{-4} \times 0.01 = 1.88 \times 10^{-6}$. For $\theta = 0.06$, D is 1.88×10^{-4}, $D\Delta\theta$ is 1.88×10^{-6}, and $\Sigma D\Delta\theta = 3.76 \times 10^{-6}$. For $\theta = 0.07$, D is 1.88×10^{-4} and $\Sigma D\Delta\theta$ would be $1.88 \times 10^{-4} \times 0.01 + 3.76 \times 10^{-6} = 5.64 \times 10^{-6}$. The numerator of equation [7] was simply evaluated by determining the value of $\Sigma D\Delta\theta$ corresponding to θ^{j-1}_{i-1} from the table and subtracting from it the value of $\Sigma D\Delta\theta$ corresponding to θ_i^{j-1} determined from the table. Linear interpolation was used to evaluate values falling between tabular data.

The moisture capacity, C, was evaluated from a table of θ vs. h, using θ as the argument and h as the function (table 1). The moisture content used to obtain C was *not* the same as the moisture content used to determine K by equations [6] and [7]. It was found that better results were obtained by using an estimated moisture content near the end of the time interval rather than the moisture content at the beginning of the time interval. Thus the estimation of K, to be constant over the time interval, was based on a knowledge of θ at the beginning of the time interval, whereas, the estimation of C, to be constant over the time interval, was based on empirical estimate of the θ near the end of the time interval (equation [8]).

$$\theta^{j+1}_i \text{ (estimated)} = (\theta^j_i - \theta^{j-1}_i)\, B + \theta^j_i \quad [8]$$

where the constant $B = 0.7$ or $t/(t + 3\frac{1}{3})$, whichever was greater. The constant B varied between 0.7 and 1.0.

PROCEDURE

The solution involved a stepwise evaluation of equation [4], which was then used to continue the solution for a new time interval. The process was repetitive.

The program involved the following general steps:

(a) From the known values of

$$\theta_i^{j-1},\ h_i^{j-1},\ \theta_{i-1}^{j-1}, \text{ and } h_{i-1}^{j-1} \text{ the values of } K_{i-1/2}^{j-1/2}$$

were estimated from equations [6] and [7] for $i = 1, 2, 3, \cdots n$.

(b) From the estimated values of

$$\theta_i^j \text{ (equation [8]) the values of } C_i^{j-1/2}$$

were estimated as described above for $i = 1, 2, 3, \cdots n$.

(c) The values of h_i^j were then computed from equation [4] for $i = 1, 2, 3, \cdots n$.

(d) From the table of h vs θ, the values of θ_i^j were computed for $i = 1, 2, 3, \cdots n$.

(e) The cumulative infiltration, CI, was computed as

$$(CI)^j = \sum_{i=1}^{n} \theta_i^j \Delta x - \sum_{i=1}^{n} \theta_i^0 \Delta x$$

(f) The infiltration rate was computed as

$$(I)^{j-1/2} = \frac{(h_o^{j-1} + h_o^j + 2G - h_1^{j-1} - h_1^{j-1})\, K_{1/2}^{j-1/2}}{2\Delta x}$$

(g) The time interval, Δt, was changed according to equation [5] and the total time was computed.

(h) The values of

$$h_i^j,\ \theta_i^j,\ (CI)^j \text{ and } (I)^{j-1/2}$$

were printed for $i = 0, 1, 2, \cdots n$.

(i) The process was then repeated starting with step "a" for as long as desired.

The value of h was assumed to be continuous (smooth) across the boundary and the water flow leaving one side of the boundary was assumed equal to that entering the other side. The distance increment, Δx, was adjusted so that the boundary between the two soils was at some "$i + \frac{1}{2}$" depth. The conductivity at the boundary was computed from the following equation:

$$K \text{ (boundary)} = 1/2\ \frac{K_a K_b}{K_a + K_b}$$

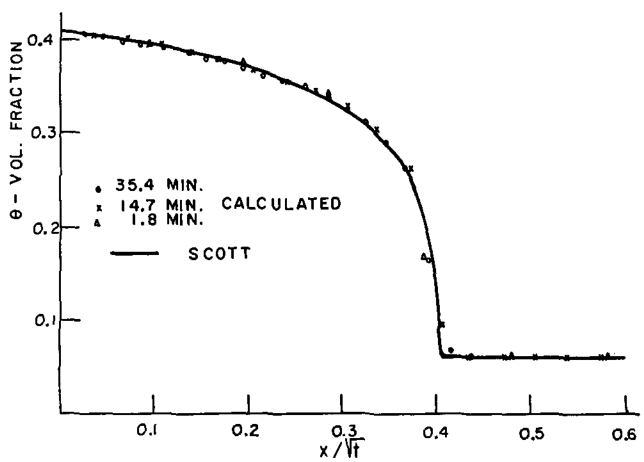

Figure 1—Comparison of the numerical method with the solution by the method of Scott et al. for estimating $\theta - x/\sqrt{t}$ for a soil with $D = \alpha e^{\beta\theta}$. $\theta = 0.061$ for $x > 0$, $t = 0$, and $\theta = 0.41$ for $x = 0$, $t > 0$. $\alpha = 1.24 \times 10^{-4}$, $\beta = 19.78$.

where K_a refers to the conductivity computed assuming the soil "above" the boundary was continuous across the boundary, and K_b refers to the conductivity computed assuming the soil "below" the boundary was continuous across the boundary.

Table 1 gives the data used for results computed with Sarpy loam and Geary silt loam. The values of the diffusivity, D, a function of moisture content, θ, were determined by the method of Bruce and Klute (1). While it is realized that the values are not precise, they are certainly within reason. For the purpose of this investigation, the data represent ranges to be found in practice, which is all that is necessary.

The relationship of θ to h shown in table 1 was determined by the usual pressure plate-membrane methods for outflow.

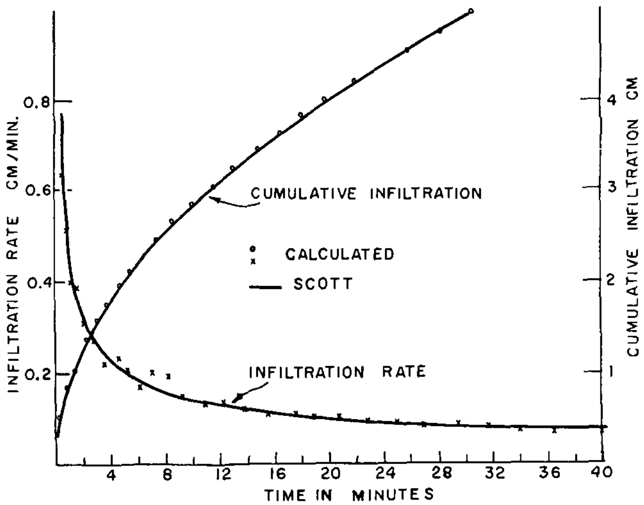

Figure 2—Comparison of the numerical method with the solution by the method of Scott et al. for infiltration rate and cumulative infiltration for a soil with $D = \alpha e^{\beta\theta}$.

Table 1—Relation of moisture content, θ, to moisture diffusivity D, and pressure head, h.

θ	Sarpy loam		Geary silt loam	
Vol. fraction	D	h	D	h
	Cm.^{2}/sec.	Cm. of H_2O	Cm.^{2}/sec.	Cm. of H_2O
0.05	1.88×10^{-4}	-6975	-	-
0.06	1.88×10^{-4}	-3365	-	-
0.07	1.88×10^{-4}	-2120	-	-
0.08	3.40×10^{-4}	-1255	-	-
0.09	7.94×10^{-4}	- 680	-	-
0.10	8.26×10^{-4}	- 447	-	-
0.11	1.30×10^{-3}	- 382	-	-
0.12	1.36×10^{-3}	- 330	-	-
0.13	1.37×10^{-3}	- 289	-	-
0.14	1.88×10^{-3}	- 259	-	-
0.15	1.91×10^{-3}	- 233	-	-
0.16	2.07×10^{-3}	- 209	-	-
0.17	2.35×10^{-3}	- 187	-	-
0.18	2.58×10^{-3}	- 168	1.6×10^{-4}	-7685
0.19	3.28×10^{-3}	- 151	3.6×10^{-4}	-5535
0.20	5.65×10^{-3}	- 134	5.6×10^{-4}	-4025
0.21	1.06×10^{-2}	- 120	7.6×10^{-4}	-3190
0.22	1.30×10^{-2}	- 106	9.5×10^{-4}	-2675
0.23	1.54×10^{-2}	- 92	1.28×10^{-3}	-2175
0.24	1.90×10^{-2}	- 78	1.92×10^{-3}	-1675
0.25	2.05×10^{-2}	- 69	2.29×10^{-3}	-1175
0.26	2.98×10^{-2}	- 64	3.61×10^{-3}	- 815
0.27	3.05×10^{-2}	- 58	4.72×10^{-3}	- 665
0.28	3.14×10^{-2}	- 53	5.31×10^{-3}	- 525
0.29	3.43×10^{-2}	- 47	7.35×10^{-3}	- 423
0.30	3.73×10^{-2}	- 43	1.02×10^{-2}	- 331
0.31	5.32×10^{-2}	- 39	1.13×10^{-2}	- 258
0.32	6.82×10^{-2}	- 34	1.22×10^{-2}	- 212
0.33	7.92×10^{-2}	- 30	1.50×10^{-2}	- 175
0.34	8.89×10^{-2}	- 26	1.78×10^{-2}	- 143
0.35	1.26×10^{-1}	- 22	2.13×10^{-2}	- 116
0.36	1.74×10^{-1}	- 18	2.74×10^{-2}	- 94
0.37	4.00×10^{-1}	- 14	2.93×10^{-2}	- 75
0.38	4.03×10^{-1}	- 10	3.32×10^{-2}	- 59
0.39	4.07×10^{-1}	- 6	3.41×10^{-2}	- 45
0.40	4.12×10^{-1}	- 3	3.85×10^{-2}	- 36
0.41	4.18×10^{-1}	0	4.25×10^{-2}	- 28
0.42	-	-	4.32×10^{-2}	- 21
0.43	-	-	4.77×10^{-2}	- 15
0.44	-	-	4.82×10^{-2}	- 10
0.45	-	-	4.86×10^{-2}	- 5
0.46	-	-	4.86×10^{-2}	0

Thus, they do not strictly apply to infiltration because of hysteresis effects, but here again, the need is only for reasonable data to check the numerical methods so that they are probably sufficient.

To evaluate the numerical methods it was thought necessary to have some independent mathematical methods by which comparisons could be made. These independent methods are available for special cases for homogeneous soils but none are known for layered soils. Consequently, the data for layered soils have not been compared with other methods of computation.

RESULTS AND DISCUSSION

Scott et al. (9) have developed a procedure whereby horizontal infiltration into a homogeneous soil can be computed for the following conditions:

$$D = \alpha e^{\beta\theta}$$
$$\theta\ (0,\ t) = \theta_1 \text{ (constant for all } t > 0).$$
$$\theta\ (x,\ 0) = \theta_2 \text{ (constant for all } x > 0 \text{ at } t = 0).$$

The data for Sarpy loam were "smoothed" so that the diffusivity was related to θ exponentially. Computations were then made for a horizontal infiltration problem for the above-stated boundary conditions. Figure 1 shows a comparison of the results computed by the numerical method with the exact solution of Scott et al. (9). For

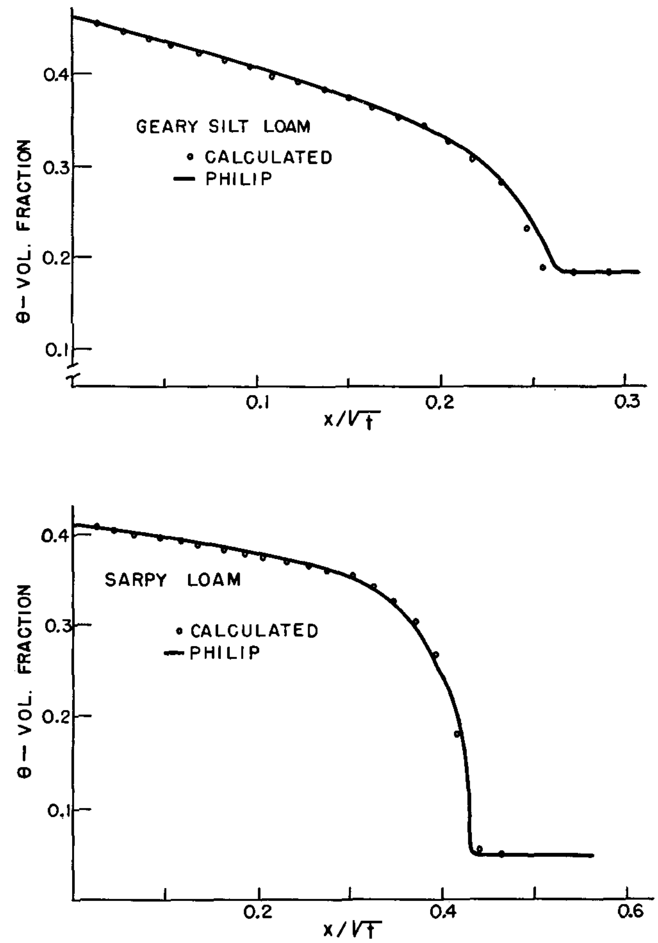

Figure 3—Comparison of the numerical method with the solution by the method of Philip for evaluating θ — $x/\sqrt{t}$ for Geary silt loam ($\theta = 0.184$ for $x > 0$, $t = 0$ and $\theta = 0.460$ for $x = 0$, $t > 0$) and Sarpy loam ($\theta = 0.050$ for $x > 0$, $t = 0$ and $\theta = 0.410$ for $x = 0$, $t > 0$).

the boundary conditions of figure 1, a plot of θ vs. $x/\sqrt{t}$ should give one curve regardless of the time. The calculated data for three different times agree quite well. The agreement between the numerical calculation and the precise method of Scott et al. (9) is also excellent. A comparison of the two methods for computing cumulative infiltration as a function of time, (given in figure 2), shows excellent agreement. However, the results for estimating infiltration rate by the numerical method are somewhat erratic.

To further check the numerical methods, calculations were made for another problem of horizontal infiltration into a semi-infinite soil at uniform initial moisture content where the actual data from table 1 were used. Philip's method (6) was used for comparison purposes. The results of this comparison for Sarpy loam and Geary silt loam show good agreement between the two methods (figure 3).

Since the method appeared to give good results on the problems where comparisons were made with other independent methods, there was reason to hope that good results could be computed for problems where the method does not require the assumptions of a uniform soil at uniform initial moisture content or a specific mathematical relationship between θ and D. However, the complications of hysteresis still limit the method to problems where the relation between moisture content and pressure head (tension) are known.

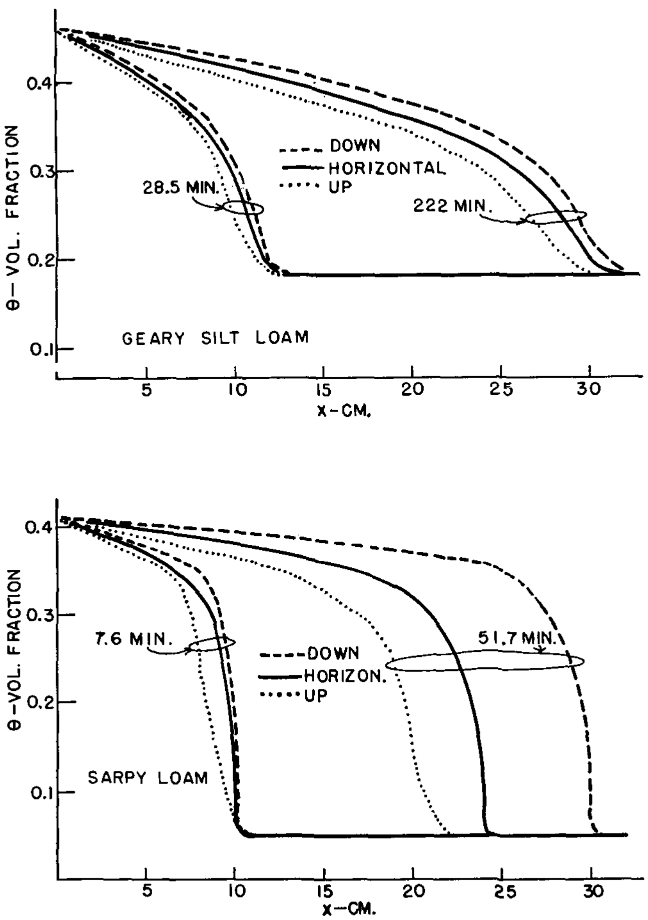

Figure 4—Effect of the gravitational term, G, on θ — x for Geary silt loam ($\theta = 0.184$ for $x > 0$, $t = 0$ and $\theta = 0.460$ for $x = 0$, $t > 0$) and Sarpy loam ($\theta = 0.05$ for $x > 0$, $t = 0$ and $\theta = 0.410$ for $x = 0$, $t > 0$).

Figure 4 shows a comparison of moisture content as a function of distance from the wet end for vertically downward infiltration, horizontal infiltration, and vertically upward infiltration (capillary rise). The computed data appear to be reasonable with the influence of the gravitational term apparent. This is especially evident in the Sarpy loam. The direction of flow appears to become increasingly important as time increases.

Figure 5 shows the comparison of moisture content and depth for a coarse soil overlaying a fine soil. Figure 5 also shows the same comparison for pressure head as the ordinate instead of moisture. The data show a moisture discontinuity at the boundary between the two soils, but, of course, there is no pressure discontinuity. Positive pressures are shown to develop in the coarse soil.

Figure 6 shows results similar to the above for a fine soil overlaying a coarse soil. Again there is a moisture discontinuity at the boundary opposite in direction from that of figure 5 but there is no pressure discontinuity.

Figure 7 shows the cumulative infiltration of the layered combinations compared to the uniform soils. The coarse-over-fine layered soil has the same cumulative infiltration curve as the coarse soil initially, since only the coarse soil is being wetted. Once the wet front reaches the boundary, the curves separate, with the cumulative infiltration decreasing for the coarse-over-fine layered soil. The comparison of the uniform fine soil with the fine-over-coarse is shown in figure 7. There is very little to distinguish between the two conditions. Apparently, water flow through the fine soil on the surface is the limiting factor.

Figure 7 also shows a comparison of infiltration rates for the same conditions. Here, also, the infiltration rate

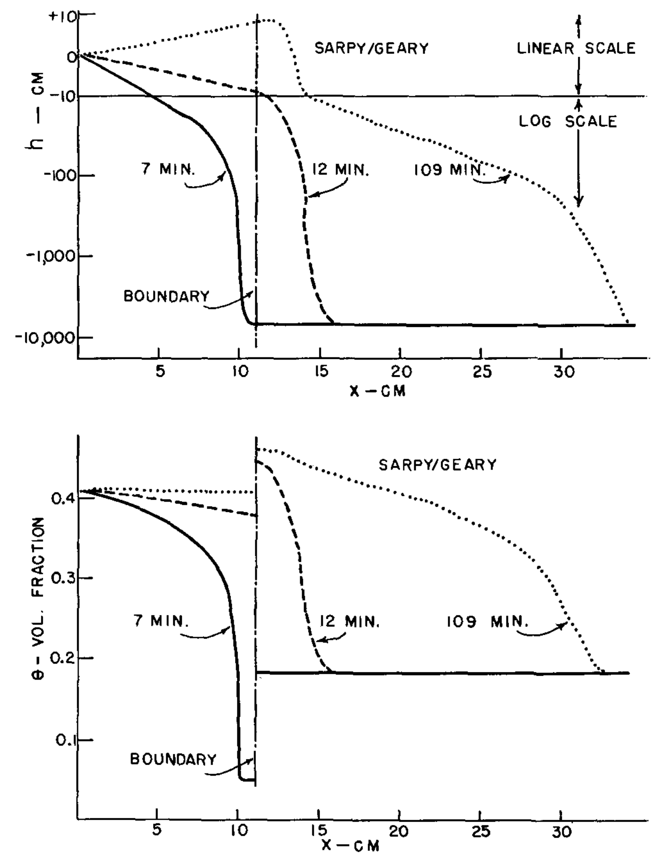

Figure 5—Relation of h — x and θ — x for infiltration into a layered soil. Sarpy over Geary with boundary at 11 cm. $\theta = 0.05$ for $0 < x < 11$ cm., $\theta = 0.184$ for 11 cm. $< x < 40$ cm. at $t = 0$. $\theta = 0.41$ for $x = 0$, $t > 0$.

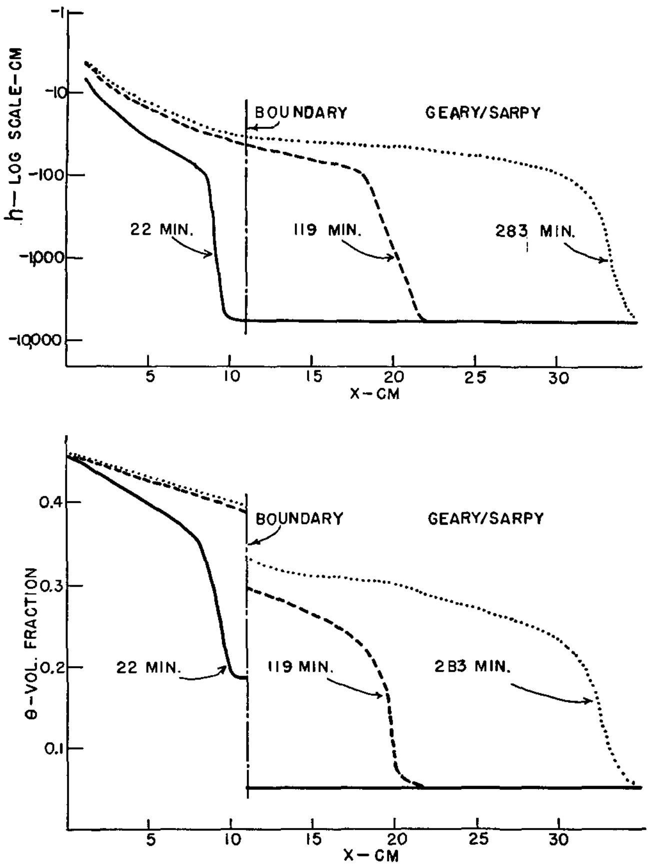

Figure 6. Relation of h — x and θ — x for infiltration into layered soil. Geary over Sarpy with boundary at 11 cm. $\theta = 0.184$ for $0 < x < 11$ cm., $\theta = 0.05$ for 11 cm. $< x < 40$ cm. at t = 0. $\theta = 0.41$ for x = 0, t < 0.

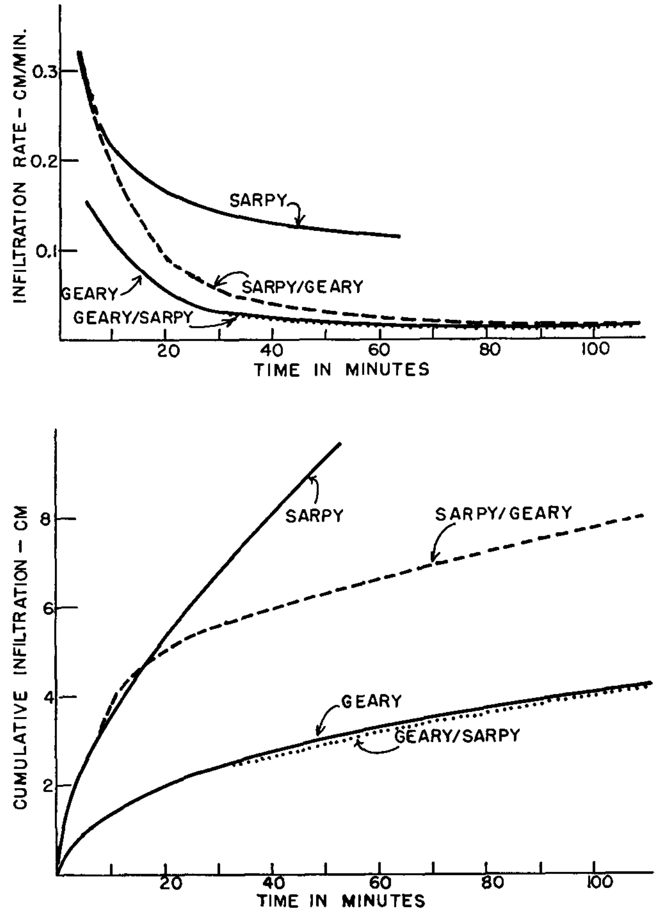

Figure 7—Influence of layer sequence on infiltration rate and cumulative infiltration.

for the coarse-over-fine layered soil was identical with the coarse soil, until the wet front reached the fine soil. The rate then fell off until it approached the infiltration rate of the uniform fine soil. The fine-over-coarse soil and the uniform fine soil gave nearly identical results throughout.

The results for the layered soils indicate that infiltration is governed by the least permeable soil layer, once the wetting front reaches this layer. The experimental results of Colman and Bodman (2) agree with the results computed herein.

SUMMARY AND CONCLUSIONS

A comparison of the numerical method with the methods of Scott et al. and Philip, for solving the moisture flow equation for horizontal infiltration into a semi-infinite soil, showed excellent agreement. Computations were made for vertical infiltration both down and up (capillary rise) for a loam and a silt loam soil.

The results appear reasonable, although no experimental evidence was attempted to verify these computations. Computations made of infiltration into layered soils indicated that infiltration was governed by flow through the least permeable soil.

The method requires, in addition to a knowledge of the initial and boundary conditions of the specific problem:

(a). A known relation between moisture content and pressure head.

(b). A known relation between moisture content and moisture diffusivity.

(c). A high speed computer for computation.

The method *does not* require that:

(a). The soil be homogeneous throughout.

(b). The soil be semi-infinite.

(c). Gravity be neglected.

(d). That the initial moisture content be uniform.

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