Return to
[illegible]
~~P. O. Box [illegible]~~
~~Ft. Collins, Colo~~
Rt. 1, Box 18
Kimberly, Idah

UNITED STATES DEPARTMENT OF AGRICULTURE
Agricultural Research Service
Soil and Water Conservation Research Division
Western Soil and Water Management Research Branch
Fort Collins, Colorado

MATHEMATICAL DERIVATIONS AS PERTAIN TO GROUND-WATER RECHARGE

by

R. E. Glover

FOREWORD

In connection with ground-water recharge, it is important that the factors affecting the build-up of ground-water mounds and their dissipation under various systems of recharge be known.

The Western Soil and Water Management Research Branch, Soil and Water Conservation Research Division, Agricultural Research Service, employed Mr. R. E. Glover, Consulting Engineer, to work out analytical solutions and to develop equations and analytical tables (with examples) pertaining to certain ground-water recharge problems.

Mr. Glover's mathematical derivations are contained in this folder.

December 1960

Spreading of the ground water mound due to recharge from a long strip of width W.

When ground water moves in one direction only the condition of continuity takes the form:

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t} \qquad \ldots .(1)$$

Where

- h represents the height of the ground water mound above the original water table level. (feet)
- x Distance measured horizontally from the center of the strip. (feet)
- t time. (seconds)

$$\alpha = \frac{KD}{V}$$

in which

- K represents the aquifer permeability (ft/sec)
- D The original saturated depth of the aquifer (feet)
- V The drainable or fillable voids expressed as a ratio to the entire volume (dimensionless)

Also let

- y represent a coordinate distance measured horizontally and at right angles to the direction of x .

A solution of equation (1) which satisfies the initial and boundary conditions

$$\begin{aligned} h &= H \quad \text{for} \quad -\frac{W}{2} < x < +\frac{W}{2} \quad \text{when} \quad t = 0 \\ h &= 0 \quad \text{for} \quad x < -\frac{W}{2} \quad \text{when} \quad t = 0 \qquad \ldots .(2) \\ h &= 0 \quad \text{for} \quad x > +\frac{W}{2} \quad \text{when} \quad t = 0 \end{aligned}$$

is

$$h = H \frac{1}{\sqrt{\pi}} \int_{u_1}^{u_2} e^{-u^2} du \qquad \ldots .(3)$$

where

$$u_2 = \frac{\left(x + \frac{W}{2}\right)}{\sqrt{4\alpha t}}$$
$$u_1 = \frac{\left(x - \frac{W}{2}\right)}{\sqrt{4\alpha t}} \quad \ldots\ldots(4)$$

Expression (3) represents the height of a ground water mound at the time t and at the distance x from the center of an infinitely long recharge area of width W lying along the axis of y. The origin of coordinates is at the center of the strip. The original height of the mound H is supposed to be due to an instantaneous recharge applied uniformly over the entire width W of the strip. Some plots of this function are shown on figure 1.

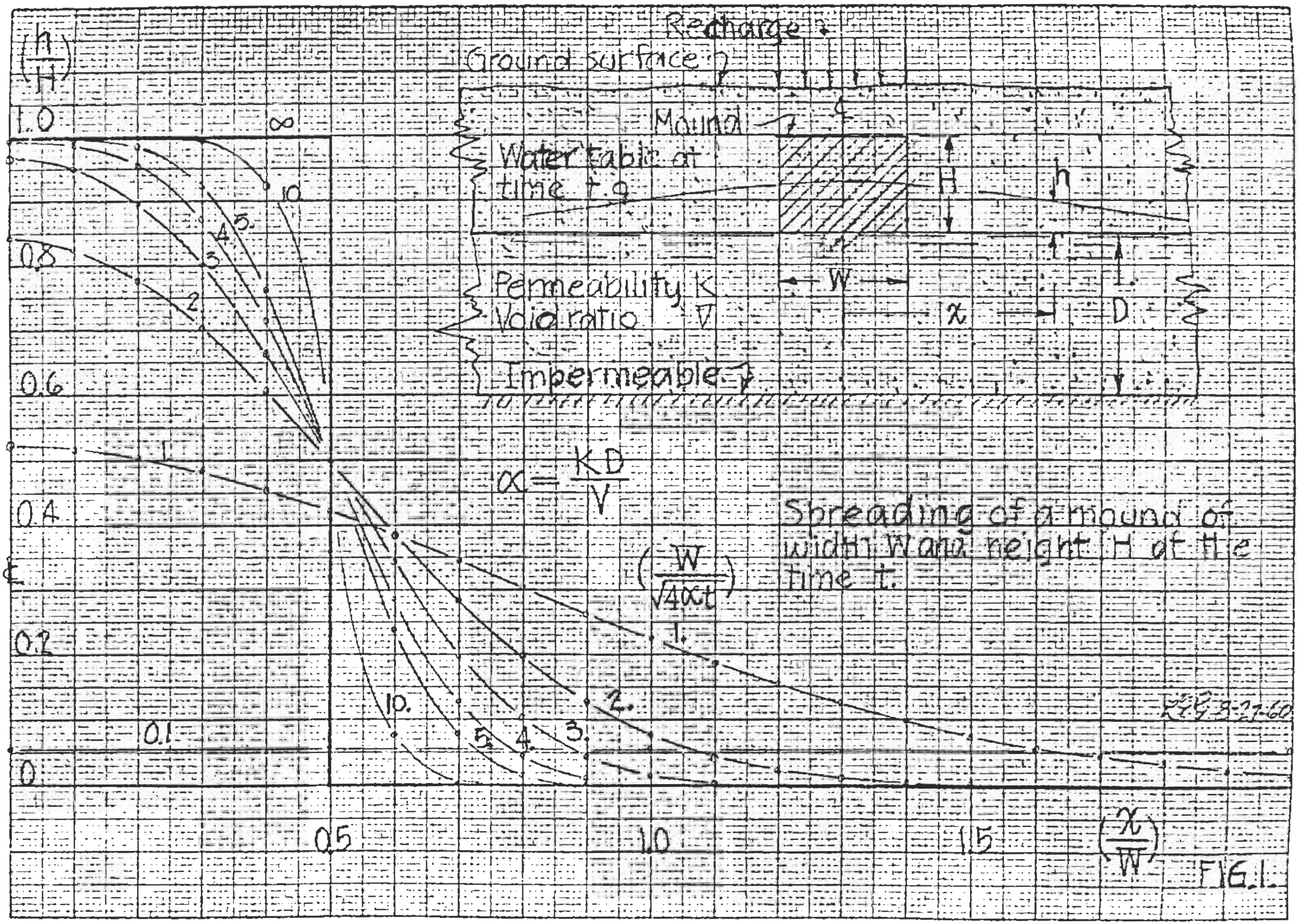
Product Law

When there is a flow of ground water in more than one direction the analytical treatment can be facilitated in certain important cases by use of the procedure to be described. This procedure employs a product of functions. It should not be confused with the older device whereby the solution of a partial differential equation is sought as a product of functions having only one independent variable. The present device employs a product of functions which have more than one independent variable. Graphical presentation of analytical data is also facilitated by the use of the product law because it avoids the difficulties presented by an excessively large number of variables. The nature and uses of the product law may be described as follows:
Suppose the function f_1 satisfies the differential equation

$$\alpha \frac{\partial^2 f_1}{\partial x^2} = \frac{\partial f_1}{\partial t} \quad \ldots\ldots(5)$$

and the function f_2 satisfies the differential equation

$$\alpha \frac{\partial^2 f_2}{\partial x^2} = \frac{\partial f_2}{\partial t} \quad \ldots\ldots(6)$$



Recharge
Ground surface
Mound
Water table at time t
Permeability K
Void ratio V
Impermeable
H
h
W
x
D
$\alpha = \frac{KD}{V}$
$\left(\frac{W}{\sqrt{4\alpha t}}\right)$
Spreading of a mound of width W and height H at the time t.
$\left(\frac{h}{H}\right)$
1.0
0.8
0.6
0.4
0.2
0
0.1
0.5
1.0
1.5
$\left(\frac{x}{W}\right)$
∞
10
5
4
3
2
1
10.
5
4
3
2.
1.

FIG. 1.

Then the product

$$h = H \quad f_1 \quad f_2 , \qquad \ldots . (7)$$

where H is a constant, will satisfy the differential equation

$$\alpha\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = \frac{\partial h}{\partial t} \qquad \ldots . (8)$$

Proof is made by substituting formula 7 into 8 and making use of the relations 5 and 6 .

In order for the solution of a differential equation to be unique it must satisfy the appropriate initial and boundary conditions. If f_1 and f_2 are unity over prescribed intervals at time zero then the product 7 will also be unity over the same intervals. When f_1 and f_2 are zero at certain boundaries the product 7 will also be zero there.

Example

Suppose a square plot 330 feet on a side is recharged at the rate of one foot per day for a period of 15 days. Compute the spreading of the ground water mound at the end of the 15 day period if the aquifer properties are:

K = .00015 ft/sec $\qquad \alpha = \frac{KD}{V}$ = 0.1 ft^2/sec

D = 100 feet

V = 0.15 (dimensionless)

If the recharge were instantly applied the height of the ground water mound would be $\frac{15}{0.15}$ = 100 feet but with the application over a period of 15 days the mounds due to the first water increments applied would have a chance to flatten before the end of the 15 day period. To approximate the effect of the gradual application the 15 day period will be divided into 5 intervals of 3 days each and it will be assumed that the increment will be applied at the middle of each interval. The addition of a 3 foot depth of recharge would create a mound having an initial height H of $\frac{3}{0.15}$ = 20 feet.

At the end of the 15 day period the height h of the mound due to the first increment applied to a strip of 330 foot width in the x direction but of infinite length in the y direction would be computed in the following way:

$t = (13.5)(86400) = 1,166,400$ seconds $\qquad \alpha = 0.1\ ft^2/sec$

$4\alpha = 0.4\ ft^2/sec \qquad \sqrt{4\alpha t} = 683 \qquad \frac{W}{2} = 165$ feet

x feet	$(x + \frac{W}{2})$	$(x - \frac{W}{2})$	u_1	u_2	$\frac{2}{\sqrt{\pi}}\int_0^{u_2} e^{-u^2} du$ *	$\frac{2}{\sqrt{\pi}}\int_0^{u_1} e^{-u^2} du$ *	$\frac{h}{H}$	h feet
0	165	- 165	.241	-.241	.267	-.267	.267	5.34
100	265	- 65	.388	-.095	.417	-.107	.262	5.24
165	330	0	.483	0	.505	0	.252	5.04
200	365	+ 35	.534	+.051	.550	.058	.246	4.92
300	465	+ 135	.682	.198	.665	.221	.222	4.44
400	565	+ 235	.827	.344	.758	.373	.192	3.84
500	665	+ 335	.974	.490	.832	.512	.160	3.20
750	915	+ 585	1.340	.857	.942	.774	.084	1.68
1000	1165	+ 835	1.707	1.222	.984	.916	.034	0.64
2000	2165	+1835	3.170	2.685	1.000	1.000	.000	0.00

The spreading of a ground water mound in the y direction due to recharge applied to a 330 foot wide strip of infinite length in the x direction would also be as shown in the above table if x were replaced by y . Then the height of the mound along a section passing through the centers of opposite sides of a square recharge plot 330 feet on a side would, by the product law, be obtained by multiplying the figures of the above table by the (h/H) value for a long strip where y = 0 . This value is 0.267 . Then if an origin is placed under the center of the square recharge plot and the axis passes through the middle of the east side of the plot, the heights of the ground water mound along the x axis would be:

x feet	$\left(\frac{h}{H}\right)_x$	$\left(\frac{h}{H}\right)_{y=0}$	$\frac{h}{H} = \left(\frac{h}{H}\right)_x \cdot \left(\frac{h}{H}\right)_{y=0}$	h feet
0	.267	.267	.0715	1.43
100	.252		.0702	1.40
165	.252		.0676	1.35
200	.246		.0659	1.32
300	.222		.0595	1.19
400	.192		.0514	1.03
500	.160		.0428	0.86
750	.084		.0225	0.45
1000	.034		.0091	0.18
2000	.000	.267	.0000	0.00

* Tables of the probability integral generally give this form. When using it a division by 2 is required in computing $\frac{h}{H}$.

A table of $\frac{1}{\sqrt{\pi}}\int_0^{u_1} e^{-u^2} du$ would simplify this computation.

The heights of the ground water mound along the northwardly running y axis would be the same as the values computed above. The heights along a line running northeasterly from the center of the plot through a corner would be computed by the product law in the following way:

x feet	$\left(\frac{h}{H}\right)_x$	y feet	$\left(\frac{h}{H}\right)_y$	$\frac{h}{H} = \left(\frac{h}{H}\right)_x \cdot \left(\frac{h}{H}\right)_y$	h feet
0	.267	0	.267	.0713	1.43
100	.262	100	.262	.0688	1.38
165	.252	165	.252	.0636	1.27
200	.246	200	.246	.0606	1.21
300	.222	300	.222	.0492	0.98
400	.192	400	.192	.0368	0.74
500	.160	500	.160	.0256	0.51
750	.084	750	.084	.0071	0.14
1000	.034	1000	.034	.0012	0.02
2000	.000	2000	.000	.0000	0.00

The height of the mound at the end of the 15 day period would be obtained by computing the contributions of each of the five increments in the way described and adding the five results. The process will be illustrated by computing the height of the mound at the center, due to the five increments, at the end of the 15 day period.

Average life of increment days	Average life of increment seconds	$\sqrt{4\alpha t}$	u_1	u_2	$\frac{2}{\sqrt{\pi}}\int_0^{u_1} e^{-u^2}du$	$\frac{2}{\sqrt{\pi}}\int_0^{u_2} e^{-u^2}du$	$\frac{1}{\sqrt{\pi}}\int_{u_1}^{u_2} e^{-u^2}du$	$\left(\frac{h}{H}\right)$ $\left(\frac{h}{H}\right)_{x=0}\cdot\left(\frac{h}{H}\right)$
1.5	129600	227	.727	-.727	.6961	-.6961	.6961	.48
4.5	388800	394	.419	-.419	.4465	-.4465	.4465	.19
7.5	648000	509	.324	-.324	.3532	-.3532	.3532	.12
10.5	907200	624	.264	-.264	.2911	-.2911	.2911	.08
13.5	1166400	683	.241	-.241	.2668	-.2668	.2668	.071
						Total		0.96

Then the height of the mound at the center after 15 days of recharge, due to all of the increments, is:

$$(20)(0.9647) = 19.3 \text{ feet}$$

Check

For a continuously applied recharge over a circular area it is possible to compute directly the height of the ground water mound under the center of the area at the time t. In the present case if the square area 330 feet on a side is assimilated to a circular area of 181 feet radius, the height of the mound at the end of 15 days of recharge comes out to be 20.0 feet. This compares well with the 19.3 feet obtained above.

Robert E. Glover
Aug 30 1960.

Spreading of a circular ground water mound

If the recharge area is circular in form a circular ground water mound will be produced. The spreading of such a mound will be the subject of this investigation. The results obtained will be useful as an idealization of many practical cases and as an independent check on other developments.

The continuity condition for the radially symmetrical case is:

$$\alpha\left(\frac{\partial^2 h}{\partial r^2}+\frac{1}{r}\frac{\partial h}{\partial r}\right)=\frac{\partial h}{\partial t} \qquad \ldots . (9)'$$

Where

- h represents the height of the ground water mound above the water table level. (feet)
- r The radial distance from the center of the recharge area. (fee
- t time.

$\alpha = \frac{KD}{V}$.

in which

- K represents the permeability of the aquifer. (ft/sec)
- D The original saturated depth of the aquifer. (feet)
- V The ratio of drainable or fillable voids to the gross volume. (dimensionless).

In addition let

- a represent the radius of the circular recharge area. (feet)
- ρ a radius running between o and a . (feet)

A solution which satisfies the conditions

$$\begin{array}{llll} h = H & \text{for } 0 < r < a & \text{when} & t = 0 \\ h = 0 & \text{for } r > a & \text{when} & t = 0 \end{array} \qquad \ldots . (10)$$

is:

$$h = H\left(\frac{1}{2\alpha t}\right)\int_0^a \rho\, e^{-\frac{r^2+\rho^2}{4\alpha t}}\, I_0\left(\frac{r\rho}{2\alpha t}\right) d\rho \qquad \ldots . (11)'$$

This relationship also finds application in Physics and has been tabulated* in the form, in the present notation,

$$P\left(\frac{a}{\sigma},\frac{r}{\sigma}\right)=\frac{e^{-\frac{r^2}{2\sigma^2}}}{\sigma^2}\int_0^a \rho e^{-\frac{\rho^2}{2\sigma^2}} I_o\left(\frac{r\rho}{\sigma^2}\right)d\rho \quad \ldots\ldots (12)$$

Where $\sigma=\sqrt{2\alpha t}$

and I_o represents the modified Bessel function of the first kind and order zero.

At the center, where $r = o$, $I_o(0) = 1.0$ and the integral can be evaluated by simple means. When this is done it is found that:

$$P\left(\frac{a}{\sigma},\frac{0}{\sigma}\right)=\left[1-e^{-\frac{a^2}{4\alpha t}}\right] \quad \ldots\ldots (13)$$

This may be written

$$P\left(\frac{a}{\sigma},\frac{0}{\sigma}\right)=\left[1-e^{-\frac{a^2}{2\sigma^2}}\right] \quad \ldots\ldots (14)$$

The tables give values of

$$P^*=\frac{P\left(\frac{a}{\sigma},\frac{r}{\sigma}\right)}{P\left(\frac{a}{\sigma},\frac{0}{\sigma}\right)} \quad \ldots\ldots (15)$$

Example

In a previous treatment of a rectangular recharge area the case of a square recharge plot 330 feet on a side was considered. The aquifer properties were assumed to be

$K = .00015$ ft/sec $\qquad \alpha = \frac{KD}{V} = 0.1$ ft^2/sec

$D = 100$ feet $\qquad V = 0.15$ (dimensionless)

* Some applications in Physics of the P function., by Joseph I. Masters., Journal of Chemical Physics. Vol 23 No.10 October 1955.

The square area will be assimilated to a circle of 181 feet radius and the spreading of a mound, originally 20 feet high, will be computed after 13.5 days have elapsed. Then:

$a = 181$ feet $\qquad t = (13.5)(86400) = 1166400$ seconds

$H = 20$ feet $\qquad \sqrt{2\alpha t} = \sqrt{233280} = 482$ feet

$$\frac{a}{\sqrt{2\alpha t}} = 0.375$$

r (feet)	$\frac{r}{\sqrt{2\alpha t}}$	P	P *	h feet
0	0	.0680	1.000	1.36
100	.207	.0664	.977	1.33
165	.342	.0642	.945	1.28
200	.415	.0608	.895	1.22
300	.623	.0564	.830	1.03
400	.830	.0491	.722	0.98
500	1.038	.0404	.595	0.81
750	1.558	.0215	.316	0.43
1000	2.075	.0085	.125	0.17
2000	4.150	.0000	.000	0.00

The value of $P\left(\frac{a}{\sigma}, \frac{0}{\sigma}\right)$ is first found from table I or from formula (14). This is the underlined value. Values of P * are then read from table II. The P column can then be completed by using the underlined value and the P * values. The heights h can then be computed by multiplying the P values by H. These h values will be found to compare reasonably well with those obtained from the square plot computation. In both cases the computations were made with a slide rule.

<u>Continuous recharge</u>

If water is recharged over the area of radius a at a rate that would cause the water table to rise at the rate R in the absence of spreading, the height of the mound at the center of the plot h_o can be obtained by integrating formula (13) with respect to time. This integration yields the formula

$$h_o = Rt\left[1 - e^{-u_1} + u_1\int_{u_1}^{\infty}\frac{e^{-u}}{u}\,du\right] \quad \ldots . (16)$$

where

$$u_1 = \frac{a^2}{4\alpha t} \quad \ldots . (17)$$

The integral which appears in this expression is known as the exponential integral. It has been extensively tabulated*.

Example

For the circular plot of 181 feet radius recharged continually at the rate of 1 foot of water per day for 15 days the product Rt would be $\frac{15}{0.15} = 100$ feet

with

$$a = 181 \text{ feet} \qquad u_1 = \frac{a^2}{4\alpha t} = \frac{181^2}{518400} = 0.0632$$

$$t = 1296000 \text{ seconds} \qquad 4\alpha t = 518400 \text{ ft}^2$$

$$= 0.1 \text{ ft}^2/\text{sec}$$

From tables

$$e^{-u_1} = 0.9387$$

$$\int_{u_1}^{\infty} \frac{e^{-u}}{u} du = 2.249 \qquad u_1 \int_{u_1}^{\infty} \frac{e^{-u}}{u} du = 0.1422$$

Then

$$1 - e^{-u_1} + u_1 \int_{u_1}^{\infty} \frac{e^{-u}}{u} du = 1 - 0.939 + .142 = 0.203$$

And

$$h_o = (100)(0.203) = 20.3 \text{ feet}$$

The line source

This idealization is useful for estimating the spreading of a ground water mound produced by a continuous recharge from canal seepage or from water released down a previously dry stream channel.

A solution of equation (1) which meets the conditions

$$\text{When } x = 0 \quad -2KD \frac{\partial h}{\partial x} = q_1 \quad \text{for} \quad t > 0$$

$$\text{When } t = 0 \quad h = 0 \quad \text{for} \quad x > 0 \qquad \ldots\ldots (18)$$

* See National Bureau of Standards Tables MT5 and MT6- Tables of Functions- Jahnke abd Emdo. Mathematical Tables- Dwight.

is:

$$h = \frac{q_1 \; x}{2 \pi \; KD} \sqrt{\pi} \int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u^2} du \quad \ldots\ldots (19)$$

Valid for $x > 0$

At $x = 0$ this solution becomes indeterminate. The height of the ground water mound is here

$$h_o = \frac{q_1 \sqrt{4 \pi \alpha t}}{2 \pi \; KD} \quad \ldots\ldots (20)$$

The recharge rate q_1 used in these expressions represents a continuous inflow from a unit length of canal or watercourse. One half of the recharge goes to each side. The quantity q_1 is specified in terms such as cubic feet per second per foot of canal. The mound is symmetrical with respect to the origin at $x = 0$.

An integration by parts yields an expression for evaluating the integral of equation (19)* in terms of the tabulated Probability integral and the tabulated exponential functions. This expression is:

$$I_{22} = \sqrt{\pi} \int_{w}^{\infty} \frac{e^{-u^2}}{u^2} du = \pi \left[\frac{e^{-w^2}}{w\sqrt{\pi}} - 1 + \frac{2}{\sqrt{\pi}} \int_{0}^{w} e^{-u^2} du \right] \quad \ldots\ldots (21)$$

Example

Recharge is obtained by ponding water in a previously dry watercourse. The average width of the bed is 80 feet and the recharge rate is found to be o.5 foot per day over the inundated area. Compute the shape of the ground water mound after the recharge operation has been carried on for 60 days. The aquifer properties are, as before:

$K = .00015$ ft/sec $\qquad \alpha = \frac{KD}{V} = 0.1 \; ft^2/sec$

$D = 100$ feet

$V = 0.15$ (dimensionless) $\qquad \pi = 3.14159+$

$\sqrt{\pi} = 1.77245$

*This integral has been tabulated by Mr. M.W. Bittinger of Colorado State University. This table may be found in the "Source Material for a course in Transient Ground Water Hydraulics" compiled by R.E. Glover and M.W. Bittinger.

Solution:

$$q_1 = \frac{(0.5)(80)}{86400} = .000463 \text{ ft}^2/\text{sec}$$

Under the stream channel where $x = 0$ the height of the mound may be computed by use of formula (20)

with

$$t = (60)(86400) = 5,184,000 \text{ seconds}$$

then

$$h_o = \frac{(.000463) \sqrt{4 \pi (0.1) 5184000}}{2 \pi (.00015)(100)} = \frac{(.000463)(2550)}{.09425}$$

$$= 12.54 \text{ feet}$$

At a point 1000 feet away from the channel

$$x = 1000 \text{ feet}$$

$$\sqrt{4 \alpha t} = \sqrt{(4)(0.1)(5184000)} = 1440 \text{ feet}$$

Then

$$\frac{x}{\sqrt{4 \alpha t}} = \frac{1000}{1440} = 0.695$$

It will be necessary to evaluate the integral I_{22}

with

$W = 0.695$ $\quad W^2 = 0.482$ $\quad$ From tables

$$e^{-w^2} = 0.6175 \qquad \frac{2}{\sqrt{\pi}}\int_0^w e^{-u^2} du = 0.6743$$

Then, from formula (21)

$$\pi\left[\frac{e^{-w^2}}{w\sqrt{\pi}} - 1 + \frac{2}{\sqrt{\pi}}\int_0^w e^{-u^2} du\right] = \pi\left[0.501 - 1 + 0.674\right] = 0.550$$

and from formula (19)

$$h = \frac{(.000463)(1000)(0.550)}{(6.2832)(.015)} = 2.70 \text{ feet}$$

The pumped well

A solution of equation (9) subject to the conditions:

$$s = 0 \quad \text{when} \quad t = 0 \quad \text{for} \quad r > 0$$

$$-2\pi KD \frac{\partial s}{\partial x} \rightarrow Q \quad \text{as} \quad r \rightarrow 0, \quad \text{for} \quad t > 0$$

is

$$s = \frac{Q}{2\pi KD} \int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du \quad \ldots . (22)$$

In this formula s represents the drawdown produced by pumping the well at the constant rate Q. The drawdown is measured from the position of the water table before pumping began. The other notation has been defined previously.

The integral appearing in this expression is a form of the tabulated Exponential Integral. Since the substitution of variable

$$v = u^2$$

changes the integral to the form

$$I_{21} = \int_{w}^{\infty} \frac{e^{-u^2}}{u} du = \frac{1}{2} \int_{w^2}^{\infty} \frac{e^{-v}}{v} dv = -\frac{1}{2} Ei(-w^2) \quad \ldots . (23)$$

The integral I_{21} has also been tabulated*

---

* Heat conduction by L.R. Ingersoll, O.J. Zobel and A.C. Ingersoll., McGraw Hill 1948. Studies of Ground Water Movement., Bureau of Reclamation Technical Memorandum No. 657- Denver, Colorado- March 1960.

The Exponential Integral has been tabulated* as the function E_1.

In this notation

$$I_{21} = \int_{w}^{\infty} \frac{e^{-u^2}}{u} du = \frac{1}{2} E_1(w^2) \quad \ldots . \quad (24)$$

Example

Suppose a well is pumped at the rate of 250 gallons per minute and the aquifer properties are, as before

$K = .00015$ ft/sec $\qquad = \frac{KD}{V} = 0.1$ ft^2/sec

$D = 100$ feet

$V = 0.15$

Compute the drawdown s at a distance of 500 feet from the well after pumping has continued for 3 days.

Solution

To convert gallons per minute to cubic feet per second multiply by .002228 then

$Q = 0.557$ ft^3/sec

One day is 86400 seconds. Then

$t = (86400)(3) = 259200$ seconds.

$\sqrt{4 \alpha t} = \sqrt{1036800} = 1018$ feet

$\frac{x}{\sqrt{4 \alpha t}} = \frac{500}{1018} = 0.492 \qquad (0.492)^2 = 0.242$

From a table of the Exponential Integral

$I_{21} = \frac{1}{2} (1.070) = 0.535$

---

* Tables of Functions and Zeros of Functions., National Bureau of Standards, Applied Mathematics Series 37.

Then, from formula(22)

$$S = \frac{(0.557)(0.535)}{(6.2832)(.015)} = 3.17 \text{ feet}$$

The assumed flow is about all this aquifer would yield to a well since at $r = 5$ the drawdown would be about 25 feet. At 1 foot from the well $\frac{x}{\sqrt{4 \alpha t}} = .001$ approximately $I_{21} = 6.62$

and

$$S = \frac{(0.557)(6.62)}{(6.2832)(0.15)} = 39.1 \text{ feet}$$

At this distance from the well there would be strong vertical components in the flow pattern. Since these were neglected in developing the formulas described herein this latter computation would not be exact. It should be of the right order of magnitude however.

Method of Images

The usefulness of such solutions as those for the Line Source and for the pumped well can be extended by the method of images since this method permits certain boundary conditions to be met which would be troublesome if this method were not available. The method is readily understood as an example will show.

Example

Suppose the pumped well of the example of the previous paragraph was located at a distance of 300 feet from a line along which the aquifer terminates against an impermeable formation. It is obvious that the radially symmetrical drawdown pattern which this well would produce in an infinitely extended aquifer would be distorted by the presence of the barrier. The condition imposed by the barrier is that there will be no flow across it. This condition can be met if the infinite aquifer is retained, a line is drawn upon it representing the position of the barrier and an image well is placed across the line from the pumped well. To meet the boundary condition the image well must be a duplicate of the pumped well, must be at the same distance from the line and must be placed directly opposite the pumped well. Superposition of the drawdown patterns of the two wells operating in the infinite aquifer will reproduce the required condition along the barrier and the proper modification of the drawdown pattern on the pumped well side. The drawdown at the barrier directly opposite the pumped well will be, for example, twice what it would be if the barrier were absent. At the barrier after three days of pumping

$$\frac{x}{\sqrt{4 \alpha t}} = \frac{300}{1018} = 0.295$$

From tables

$$I_{21} = 0.975$$

Then for the pumped well

$$S = \frac{(0.557)(0.975)}{(6.2832)(.015)} = 5.76 \text{ feet}$$

This is the drawdown the pumped well would produce in the infinitely extended aquifer. To account for the presence of the barrier superimpose the drawdown due to the image well which is also 5.76 feet at this point. Then the total drawdown at the barrier is $5.76 + 5.76 = 11.52$ feet.

When a pumped well is near a stream which maintains the water level in the aquifer a similar ruse can be employed. In this case there is to be no drawdown at the stream. This can be arranged by using an image well which, in this case, must be a recharge well.

Extensions of this procedure will permit the treatment of a well located in the corner where two streams join or for a well located between a stream and a barrier. In the latter case an infinite succession of images is obtained by successively meeting the conditions at the two boundaries. The result is an infinite series which, however, converges rapidly.

Perching layer present- Steady state case

In the section shown on figure 2 a layer of low permeability restricts the downward flow of water and thereby creates a perched water table. If the permeability of the aquifer above the perching layer is K then the horizontal flow F through the saturated thickness h at x is given by

$$F = Kh \left(\frac{dh}{dx} \right) \quad \ldots . (25)$$

Under steady state conditions, including a unity gravity gradient through the perching layer and an additional gradient (C/m) due to a capillary tension C exerted on the bottom of the perching layer by the partly saturated sands below the perching layer. The continuity condition is:

$$+ \frac{dF}{dx} dx - P \left(\frac{h}{m} + 1 + \frac{C}{m} \right) dx = 0$$

By use of equation 25 and after rearrangement this relation takes the form:

$$h \frac{d^2h}{dx^2} + \left(\frac{dh}{dx} \right)^2 - \frac{P}{Km} h = \frac{P}{K} \left(1 + \frac{C}{m}\right) \quad \ldots (26)$$

The substitution

$$\frac{dh}{dx} = y \qquad \frac{d^2h}{dx^2} = y \frac{dy}{dh} \quad \ldots . (27)$$

transforms this expression to

$$hy \frac{dy}{dh} + y^2 = \frac{P}{Km} h + \frac{P}{K} \left(1 + \frac{C}{m}\right) \quad \ldots . (28)$$

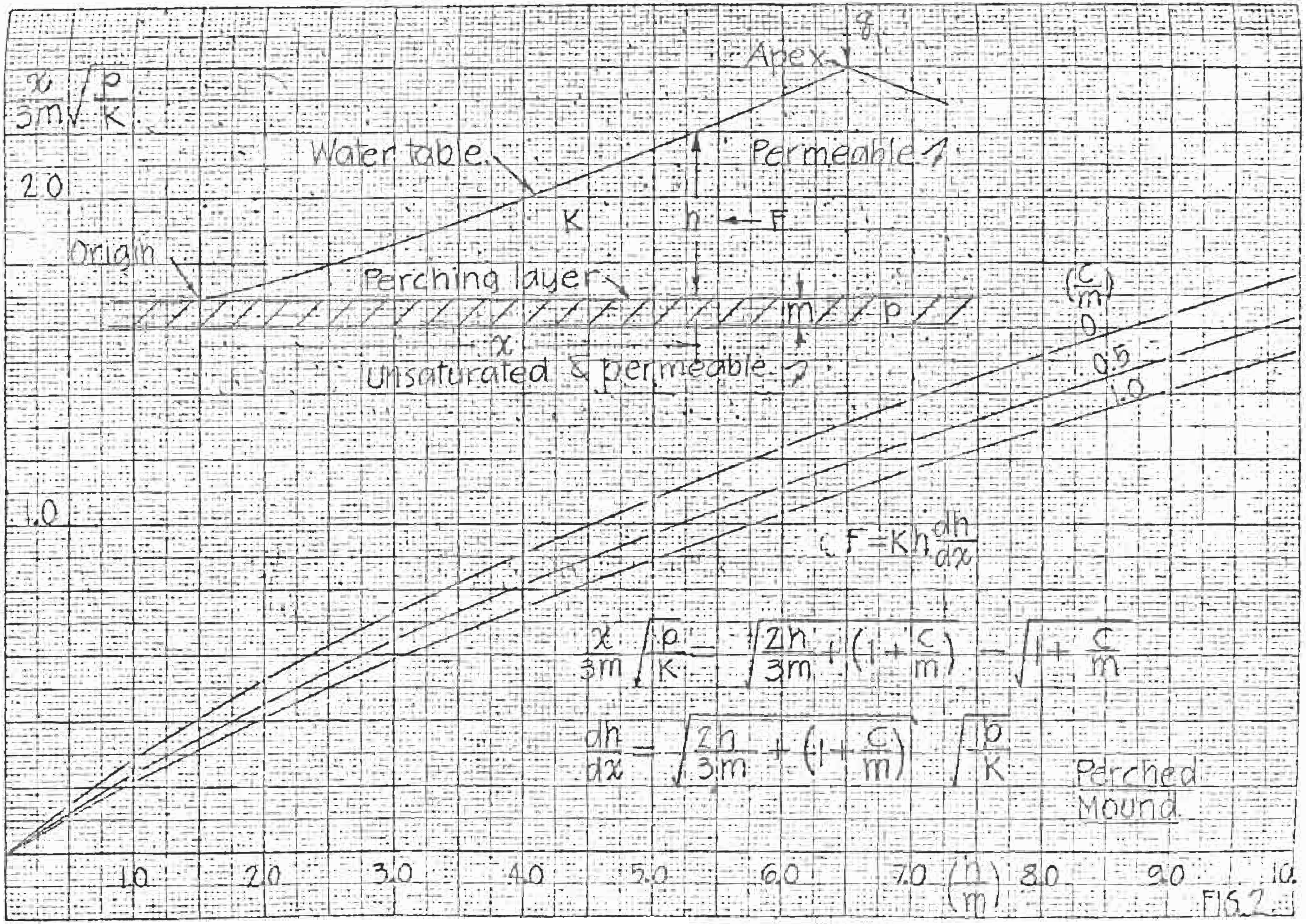
The additional substitution

$$y^2 = v \qquad 2y \frac{dy}{dh} = \frac{dv}{dh} \quad \ldots . (29)$$

Transforms it to:

$$\frac{dv}{dh} + \frac{2}{h} v = \frac{2P}{Km} + \frac{2P}{Kh} \left(1 + \frac{C}{m}\right) \quad \ldots . (30)$$

This is a linear differential equation for which an integrating factor is h^2.



$\frac{x}{3m}\sqrt{\frac{p}{K}}$
Apex
Water table
Permeable
2.0
K
h
F
Origin
Perching layer
m
p
x
Unsaturated & permeable
$(\frac{c}{m})$
0
0.5
1.0
1.0
$F = Kh\frac{dh}{dx}$
$\frac{x}{3m}\sqrt{\frac{p}{K}} = \sqrt{\frac{2h}{3m} + (1+\frac{c}{m})} - \sqrt{1+\frac{c}{m}}$
$\frac{dh}{dx} = \sqrt{\frac{2h}{3m} + (1+\frac{c}{m})}\sqrt{\frac{p}{K}}$
Perched Mound
1.0
2.0
3.0
4.0
5.0
6.0
7.0
$(\frac{h}{m})$
8.0
9.0
10.

FIG. 2

By integration

$$h^2 v = \frac{2p}{Km}\frac{h^3}{3} + \frac{2p\left(1+\frac{C}{m}\right)}{K}\frac{h^2}{2} + C_1 \qquad \ldots . (31)$$

or

$$h^2\left(\frac{dh}{dx}\right)^2 = \frac{2p}{Km}\frac{h^3}{3} + \frac{2p\left(1+\frac{C}{m}\right)}{K}\frac{h^2}{2} + C_1 \qquad \ldots . (32)$$

If the origin is taken at the end of the mound where $h = 0$ then $C_1 = 0$ and, by rearrangement:

$$\int_0^h \frac{dh}{\sqrt{\frac{2h}{3m} + \left(1+\frac{C}{m}\right)}} = x\sqrt{\frac{p}{K}} \qquad \ldots . (33)$$

If the integration is performed the result is:

$$\sqrt{\frac{2h}{3m} + \left(1+\frac{C}{m}\right)} - \sqrt{1+\frac{C}{m}} = \frac{x}{3m}\sqrt{\frac{p}{K}} \qquad \ldots (34)$$

Example

Suppose a bed of permeability $K = .002$ ft/sec rests upon a perching layer of thickness $m = 15$ feet and having a permeability of $p = .000001$ ft/sec. The sand below the perching layer is permeable but is unsaturated and exerts a capillary tension of $C = 5$ feet of water. A canal traverses the area and has a seepage amounting to 1 cubic foot per second per mile of canal. Compute the ultimate height of the ground water mound.

Solution

$\frac{C}{m} = \frac{5}{15} = 0.333$ $\qquad$ $\left(1+\frac{C}{m}\right) = 1.3333$ $\qquad$ $\frac{p}{K} = .0005$

$K = 0.002$ ft/sec $\qquad$ $\sqrt{1+\frac{C}{m}} = 1.1547$ $\qquad$ $\sqrt{\frac{p}{K}} = 0.02236$

$q_1 = \frac{1.0}{5280}$ $\qquad$ $\frac{q_1}{2} = \frac{1}{10560} = .0000947$ $\qquad$ $3m\sqrt{\frac{K}{p}} = 2010$

h feet	$\frac{h}{m}$	$\frac{2h}{3m} + (1 + \frac{C}{m})$	$\sqrt{\frac{2h}{3m} + (1 + \frac{C}{m})}$	$\frac{x}{3m}\sqrt{\frac{p}{K}}$	x feet	$\frac{dh}{dx}$	F (ft^2/sec
0	0	1.3333	1.1547	0	0	.02582	0
0.2	.01333	1.3422	1.1585	.0038	7.6	.02590	.000010
0.4	.02666	1.3511	1.1624	.0077	15.5	.02602	.000020
0.6	.04000	1.3600	1.1662	.0115	23.1	.02610	.000030
0.8	.05333	1.3689	1.1700	.0153	30.8	.02620	.000041
1.0	.06666	1.3778	1.1738	.0191	38.4	.02627	.000052
1.2	.08000	1.3867	1.1776	.0229	46.0	.02632	.000063
1.4	.09333	1.3956	1.1814	.0267	53.7	.02645	.000074
1.6	.10666	1.4044	1.1851	.0304	61.2	.02650	.000084
1.8	.12000	1.4133	1.1890	.0343	69.0	.02660	.000095
2.0	.13333	1.4222	1.1925	.0378	76.0	.02670	.000106

The x distance required to make F equal to $(q_1/2)$ is found by interpolation to be 68.3 feet. This is the distance from the toe of the mound to the apex. The total width of the mound is (2)(68.3) = 136.6 feet. The height of the mound at the apex is also found by interpolation to be 1.78 feet.

As a check compute the width necessary to transmit the seepage through the perching layer under the action of the gravity gradient and the capillary tension. Then if the width from the toe to the apex is $\frac{W_1}{2}$

$$p\left(1 + \frac{C}{m}\right)\frac{W_1}{2} = \frac{q_1}{2}$$

and

$$\frac{W_1}{2} = \frac{q_1}{2p\left(1 + \frac{C}{m}\right)} = \frac{0000947}{000001333} = 70.9 \text{ feet}$$

This is greater than the computed distances by

70.9 - 68.3 = 2.6 feet. This is reasonable since the pressures due to the saturated depths h were neglected in the check computation. The check is considered satisfactory.

To get some idea of the time required to establish the mound after the seepage q_1 begins it may be assumed that the first seepage water to reach the perching layer is used to build up the volume of the mound but, as the volume of the mound grows, a part of the supply q_1 is consumed by seepage through the perching layer. When the final stable configuration is reached all of the supply q_1 is consumed by the perching layer seepage. If the time required to accumulate the volume of the mound, with no loss through the perching layer, is represented by T then if h_c and h_m represent the transient and final values of h at the apex of the mound it would be reasonable to write

$$\frac{dh_c}{dt} = \frac{h_m}{T} - \frac{h_c}{T} \qquad \ldots\ldots (35)$$

A solution which makes $h_c = 0$ when $t = 0$ is

$$h_c = h_m (1 - e^{-\frac{t}{T}}) \qquad \ldots\ldots (36)$$

The mound is nearly triangular in shape and in our case the volume of the mound per unit of length along the apex, is approximately

$$(68.3)(1.78) = 121.2 \text{ cubic feet per foot.}$$

The volume of water contained in it is

$$(121.2)(V) = (121.2)(0.15) = 18.2 \text{ cubic feet per foot.}$$

The seepage supply is $q_1 = \frac{1}{5280}$ cubic feet per second per foot of canal. Then

$$T = \frac{18.2}{q_1} = (18.2)(5280) = 96000 \text{ seconds.}$$

The time required to bring the mound to any specified part of completion may now be estimated. Suppose it is desired to know how long it would take to make $\frac{h_c}{h_m} = 0.9$

Then by rearrangement

$$e^{-\frac{t}{T}} = 1 - \frac{h_c}{h_m} = 0.1$$

From tables

$$\frac{t}{T} = 2.302$$

And

$$t = (2.302)(96000) = 221000 \text{ seconds}$$

This would be

$$\frac{221000}{86400} = 2.56 \text{ days}$$

Robert M. [illegible]
Sept 1 1960

Shape of the ground water mound beneath a long recharged strip-steady state case.

When the recharge rate i is included in equation 26 the result is:

$$h \frac{d^2h}{dx^2} + \left(\frac{dh}{dx}\right)^2 - \frac{ph}{Km} = \frac{p}{K}\left(1 + \frac{C}{m} - \frac{i}{p}\right) \quad \ldots\ldots (37)$$

The method of integration followed previously is not of much use here because of the difficulty of integrating expression (32) when C_1 is not zero. A different procedure is therefore desirable. It is to be expected from physical considerations that the greatest saturated depth will be found under the recharged strip and that there will be relatively little variation of depth in this zone. These conditions favor the development of a first approximation solution appropriate for minor departures from a saturated depth D. With this simplification equation (26) is replaced by the linear differential equation

$$\frac{d^2h}{dx^2} = \frac{p}{KD}\left(\frac{D}{m} + 1 + \frac{C}{m} - \frac{i}{p}\right) \quad \ldots\ldots (38)$$

By integration, if $\frac{dh}{dx}$ is to be zero when $x = 0$

$$\frac{dh}{dx} = \frac{p}{KD}\left(\frac{D}{m} + \frac{1+C}{m} - \frac{i}{p}\right) x \quad \ldots\ldots (39)$$

This choice places the origin of x at the middle of the strip. By integration, if $h = h_0$ when $x = 0$.

$$h = \frac{p}{KD}\left(\frac{D}{m} + 1 + \frac{C}{m} - \frac{i}{p}\right)\frac{x^2}{2} + h_0 \quad \ldots\ldots (40)$$

Example

As an example of the use of this equation it will be supposed that the canal of the previous example produces a downward percolation through a strip 60 feet wide. The depth of the mound under the center of the strip will be computed.

Solution

The infiltration rate is:

$$i = \frac{1}{(5280)(60)} = .000{,}003{,}16 \text{ ft/sec}$$

The solution is best made by trial. The depth h_0 at the center of the strip is assumed and a reasonable value of D is also selected. The head h at the edge of the strip is then computed from equation (40) and the flow at the edge of the strip is computed as $-KD\frac{dh}{dx}$ by use of equation (39). The choice is satisfactory if the depth and flow correspond to one of the entries of the table on page 20. In this case the combination $h_0 = 1.2$ feet $D = 1.00$ feet.

yields a depth at the edge of the strip of about 0.8 feet and a flow of .000042 ft^2/sec. In the table the corresponding entries are $h = 0.8$ feet and $F = .0000418$ ft^2/sec. This agreement is considered satisfactory.

Circular ground water mound-Steady state case.

When water is recharged over a circular area overlying a semi-permeable perching layer a ground water mound will be built up as shown on figure 3. If the outward flow through the cylindrical surface of radius r and height h is F then the condition for a steady state is:

$$-\frac{dF}{dr} - 2\pi rp\left(\frac{h}{m} + 1 + \frac{C}{m}\right) + i\,2\pi r = 0$$

In this expression the symbols have the following significance:

C the capillary tension exerted at the base of the perching layer, expressed as equivalent feet of water (feet)

F the flow through a cylindrical shell of radius r and height h. The flow is positive if outward (ft^3/sec)

h height of the water table above the perching layer (feet)

i the infiltration rate due to recharge over a circular area of radius a. (ft/sec)

K the permeability of the saturated material above the perching layer. (ft/sec)

m the thickness of the perching layer (feet)

p the permeability of the perching layer (ft/sec)

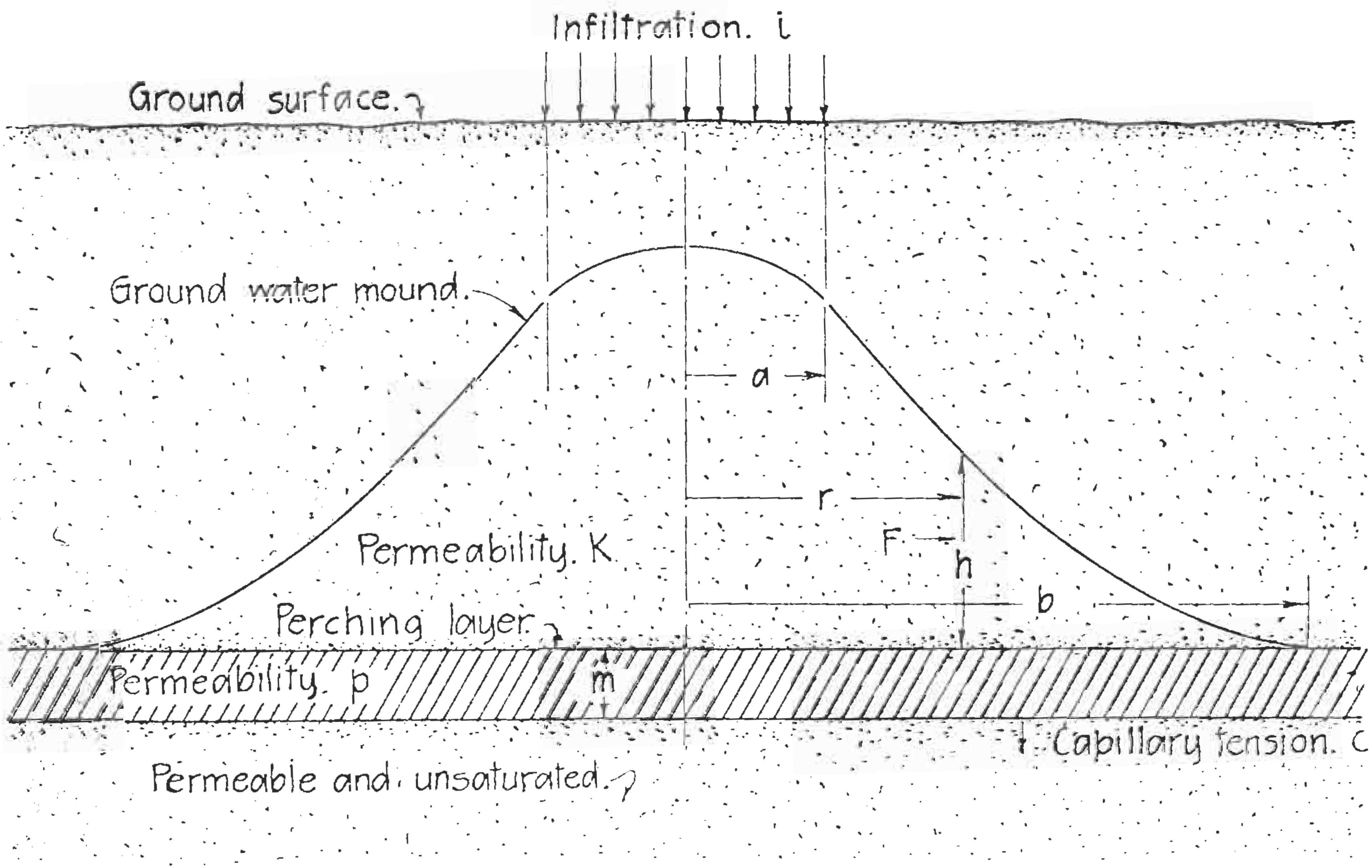
r radius from the center of the recharge area (feet)

$\pi = 3.14159+$ ....

The unit quantity in the parenthesis represents the gravity gradient through the perching layer.

If the flow is:

$$F = -K\,2\pi rh\frac{dh}{dr} \qquad \ldots\ldots (41)$$



Circular recharge area over a perching layer.

FIG.3

Then the above relation can be put in the form

$$\frac{d}{dr}\left(rh\frac{dh}{dr}\right) - \frac{pr}{K}\left(\frac{h}{m} + 1 + \frac{C}{m}\right) = -\frac{ir}{K} \quad \ldots\ldots (42)$$

Outside the recharge area i will be zero and the right hand member will be absent.

If the operations indicated in formula (42) are carried out a non-linear ordinary differential equation of the second order is obtained. This equation does not yield to the ordinary methods of solution and is not amenable to solution by series of the types commonly employed. It would probably yield to numerical methods but much computation work would be needed because of the number of variables involved and the awkward way the boundary conditions enter. The known quantities in any application would be the radius of the recharge area, the rate of recharge and the properties of the aquifer and of the perching layer. The mound will have an outer radius b which is not known but the boundary condition at the radius b is:

$$\frac{dh}{dr} = -\sqrt{\frac{p}{K}\left(1 + \frac{C}{m}\right)} \quad \ldots\ldots (43)$$

The substitution of variable

$$h = mu \qquad r = \gamma\xi \qquad \gamma = m\sqrt{\frac{K}{p}} \quad \ldots\ldots (44)$$

reduces the differential equation to the form

$$u\frac{d^2u}{d\xi^2} + \left(\frac{du}{d\xi}\right)^2 + \frac{u}{\xi}\frac{du}{d\xi} - u = \left(1 + \frac{C}{m}\right) - \frac{i}{p} \quad \ldots (45)$$

The boundary condition at the outer boundary of the region in which $i = 0$ then becomes:

$$\frac{du}{d\xi} = \sqrt{\left(1 + \frac{C}{m}\right)} \quad \ldots\ldots\ldots\ldots (46)$$

It should be possible to prepare a series of charts, each applicable for a specific value of $\left(1 + \frac{C}{m}\right)$ and each containing a family of curves, computed for selected values of ξ when $u = 0$ where the boundary condition is known. The integrations would be made backwards from these terminal values. An intersecting set of curves passing through equal values of

$$\frac{F}{2\pi Km^2} = -\xi u\frac{du}{d\xi} \quad \ldots\ldots\ldots\ldots (47)$$

could then be plotted on these charts. A solution for an actual case would be sought by finding, on the chart prepared for the appropriate value of $\left(1 + \frac{C}{m}\right)$, that curve which would yield the required flow F

at the prescribed radius a. Similar developments would apply to the zone inside the radius a.

In view of the probability that the aquifer properties and geometry will not conform exactly to the idealization of figure 3 in any actual case it may be preferable to employ approximate procedures based upon relatively simple formulas rather than to spend the considerable time expense needed to prepare such charts. Those approximate formulas are obtained by giving the quantity $\frac{h}{m}$ an average value $\frac{h_a}{m}$ in equation (42) which then becomes easily integrable. A solution of equation (42) subject to the requirement that $h = 0$ when $r = b$ is:

$$h^2 = \frac{p\, b^2 R}{2K} \left(\log_e \frac{b^2}{r^2} - 1 + \frac{r^2}{b^2} \right) \quad \ldots\ldots \quad (48)$$

where

$$R = \left(\frac{h_a}{m} + 1 + \frac{C}{m} \right) \quad \ldots\ldots \quad (49)$$

A value of b can be obtained from the relation $i \pi a^2 = pR \pi b^2$ or

$$b = a \sqrt{\frac{i}{pR}} \quad \ldots\ldots \quad (50)$$

Example

As an example of the use of these formulas suppose a circular area of radius 186 feet is recharged at the rate of 1 foot per day. A perching layer below the recharge area has a thickness of 15 feet and a permeability of .000,001 ft/sec. The sand overlying the perching layer has a permeability of .002 ft/sec. An unsaturated formation below the perching layer imposes a capillary tension of 5 feet of water. Then

a = 186 feet $\quad i = \frac{1}{86400}$ ft/sec $\quad K = 0.002$ ft/sec

m = 15 feet $\quad p$ = .000001 ft/sec $\quad C$ = 5 feet

$i \pi a^2 = 1.258$ ft3/sec $\quad (1 + \frac{C}{m}) = 1.3333$

Solution

As a first approximation take $\frac{h_a}{m} = 0$. Then

$$R = (1 + \frac{C}{m}) \qquad b = a \sqrt{\frac{i}{pR}} = 186 \sqrt{\frac{1}{.1152}} = 548 \text{ feet}$$

$$\frac{b^2}{a^2} = 8.68 \qquad \frac{a^2}{b^2} = .1152 \qquad \frac{pb^2 R}{2K} = \frac{.400}{.004} = 100$$

$$\log_e \frac{b^2}{a^2} = 2.1610 \quad \left(\log_e \frac{b^2}{a^2} - 1 + \frac{a^2}{b^2}\right) = 1.2762$$

Then from formula (48) the saturated depth h_o^2 at the radius a is:

$$h_o^2 = (100)(1.2762) = 127.62 \quad \text{and} \quad h_o = 11.28 \text{ feet}$$

It is now possible to estimate the average head on the perching layer since if the mound is idealized as having the shape of a truncated cone the average head will be

$$h_a = \frac{h_o}{3} \left(1 + \frac{a}{b} + \frac{a^2}{b^2}\right) \quad \ldots\ldots\ldots (51)$$

Then

$$h_a = \frac{11.28}{3} (1 + .339 + .1152) = \frac{(11.28)(1.454)}{3} = 5.46 \text{ feet}$$

The new value of R is:

$$R_1 = \left(\frac{5.46}{15} + 1 + \frac{5}{15}\right) = 1.70$$

As before

$$b_1 = 186 \sqrt{\frac{1}{.1469}} = 485 \text{ feet}$$

$$\frac{b_1^2}{a^2} = 6.82 \qquad \frac{a^2}{b_1^2} = .1469 \qquad \frac{p\, b_1^2\, R_1}{2\, K} = \frac{4.00}{.004} = 100$$

$$\log_e \frac{b_1^2}{a^2} = 1.9200 \quad \left(\log_e \frac{b_1^2}{a^2} - 1 + \frac{a^2}{b^2}\right) = 1.0669$$

$$h_o^2 = (100)(1.0669) = 106.69 \qquad h_o = 10.34 \text{ feet}$$

This is the estimated height of the mound at the edge of the recharge area.

It remains to treat the portion of the mound within the radius a. A solution of equation (42) which satisfies the requirements

$$\frac{dh}{dr} = 0 \quad \text{when} \quad r = 0 \quad \text{if} \quad h \neq 0$$

$$h = h_a \quad \text{when} \quad r = a \qquad \ldots\ldots\ldots \qquad (52)$$

is:

$$h^2 = h_a{}^2 + \frac{(i - pR)}{2K} (a^2 - r^2) \qquad \ldots\ldots\ldots \qquad (53)$$

Example

Compute the height of the mound under the center of the recharge area described in the previous problem.

Solution

With $R = \left(\frac{10.34}{15} + 1 + \frac{5}{15}\right) = 2.023 \qquad pR = 000{,}002{,}023$

$i = \frac{1}{86400} = 00001158 \quad (i - pR) = 000{,}009{,}56$ ft/sec

$\frac{(i - pR)}{2K} = .00239$

Then when $r = 0$.

$h^2 = 10.34^2 + (.00239)(186)^2 = 106.7 + 82.7 = 189.4$

$h = 13.76$ feet

This is the estimated height of the perched water table, above the perching layer, at the center of the mound.

The time required to establish the mound can be estimated by use equation (36).

[illegible signature]
Sept 12 1960.

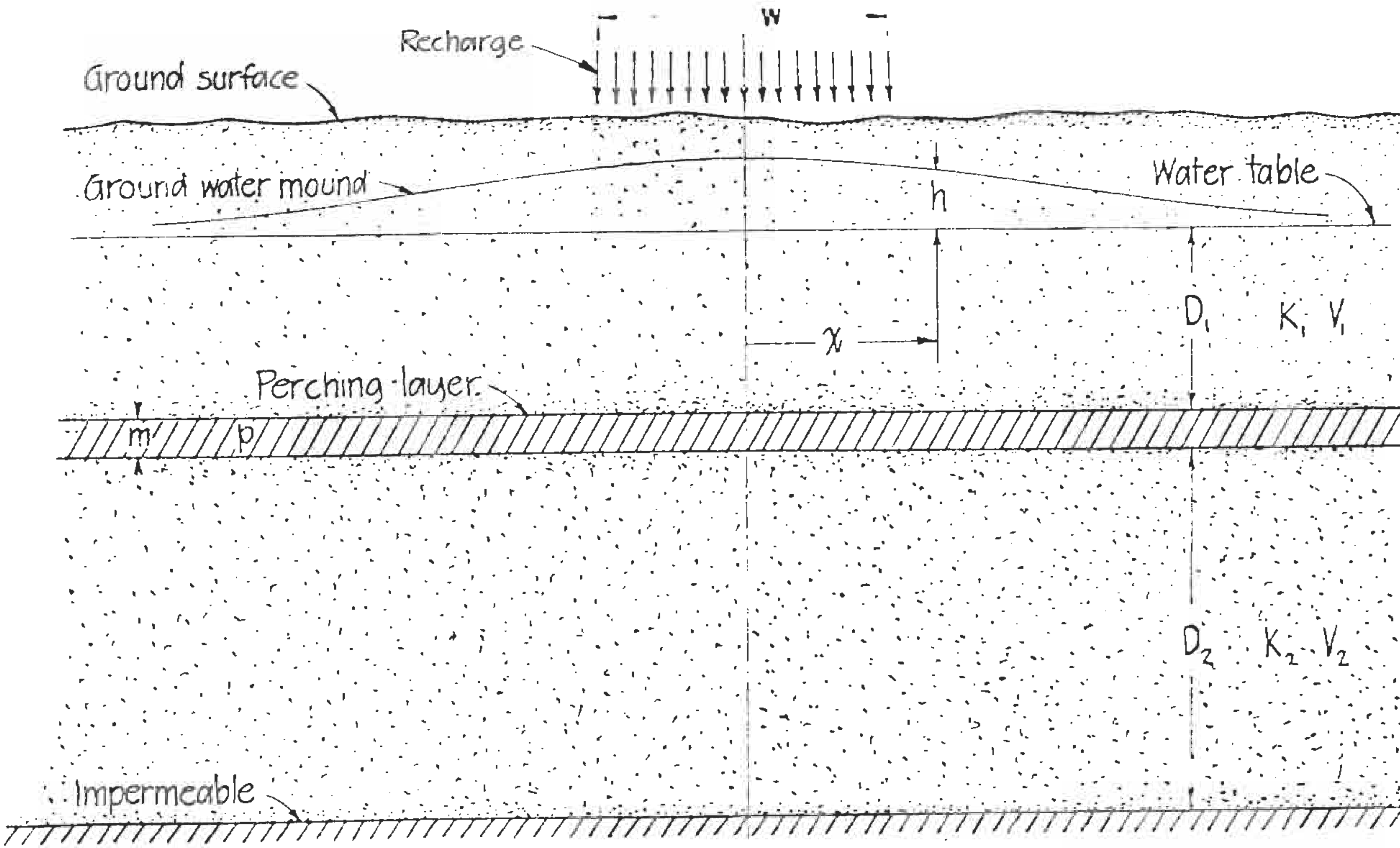
Layered soil - Criterion for the intermediate layer.

A bed of low permeability interposed between two aquifers, as shown on figure 4, will modify the spreading of a ground water mound caused by recharge over a long strip of width W. If the thickness m and permeability p of the low permeability layer, which will be referred to hereafter as the perching layer, are such that little resistance is interposed to the vertical movement of water between the upper and lower aquifers then the two aquifers will act essentially as one. On the other hand, if the perching layer strongly restricts the vertical movement of water then the upper bed will act essentially alone. It will be the purpose of this investigation to develop a criterion to indicate which of these two types of behavior may be expected in any actual case. It should be realized at the outset, however, that such a criterion can never provide a sharp distinction between the two cases because there will be intermediate cases where the perching layer neither permits free vertical movement of water nor completely restricts it. Nevertheless, a criterion of this kind will be useful to indicate the type of behavior to be expected.

To obtain such a criterion it will first be supposed that the two beds act as one. The rate of vertical flow compatible with this assumption will be computed and the head needed to drive the vertical flow through the perching layer will then be estimated and compared to the head available above the perching layer. If the perching layer consumes too much head then it will be evident that the two beds can not act together as one. If the head needed to drive the assumed vertical flow through the perching layer is large compared to the available head then it will be evident that the upper bed must act essentially alone.

If the two aquifers act together then the joint transmissibility will be the sum of the individual transmissibilities $(K_1D_1 + K_2D_2)$ and the effective voids ratio will be that of the upper aquifer V_1. The diffusion constant will be $\alpha = \dfrac{(K_1D_1 + K_2D_2)}{V_1}$. The spreading of a ground water mound caused by an instantaneous recharge of depth H over the width W of a long strip is given by formula (3). Differentiation of this formula twice with respect to x gives

$$\frac{\partial^2 h}{\partial x^2} = \frac{H}{\sqrt{\pi}}\left[-\frac{2(x+\frac{W}{2})}{(4\alpha t)^{\frac{3}{2}}}e^{-\frac{(x+\frac{W}{2})^2}{4\alpha t}} + \frac{2(x-\frac{W}{2})}{(4\alpha t)^{\frac{3}{2}}}e^{-\frac{(x-\frac{W}{2})^2}{4\alpha t}}\right] \quad \ldots\ldots\ldots (54)$$



Layered soil.

FIG. 4.

It will be convenient to work with the point $x = 0$ where

$$\left(\frac{\partial^2 h}{\partial x^2}\right)_0 = -\frac{H}{\sqrt{\pi}}\frac{2W}{(4\alpha t)^{\frac{3}{2}}} e^{-\frac{W^2}{16\alpha t}} \quad \ldots\ldots (55)$$

The vertical flow of water through the porching layer is here:

$$-K_2 D_2 \left(\frac{\partial^2 h}{\partial x^2}\right)_0 = +\frac{H\,2K_2 D_2 W}{\sqrt{\pi}(4\alpha t)^{\frac{3}{2}}} e^{-\frac{W^2}{16\alpha t}} \quad \ldots\ldots (56)$$

The head which would be required to drive this flow through the perching layer would be

$$-\frac{K_2 D_2 m}{p}\left(\frac{\partial^2 h}{\partial x^2}\right)_0 = +\frac{H m\,2K_2 D_2\,8}{p\sqrt{\pi}\,W^2}\left(\frac{W}{2\sqrt{4\alpha t}}\right)^3 e^{-\left(\frac{W}{2\sqrt{4\alpha t}}\right)^2} \quad \ldots\ldots (57)$$

The function

$$\frac{2}{\sqrt{\pi}}\left(\frac{W}{2\sqrt{4\alpha t}}\right)^3 e^{-\left(\frac{W}{2\sqrt{4\alpha t}}\right)^2}$$

reaches a value of 0.4618 when $\frac{W}{2\sqrt{4\alpha t}} = 1.2$.

This is near the maximum value. For the same value of the parameter the head at $x = 0$ is:

$$h_0 = H\frac{2}{\sqrt{\pi}}\int_0^{\frac{W}{2\sqrt{4\alpha t}}} e^{-u^2}du = H\ 0.9103$$

Then the ratio of the assumed head loss through the perching layer h_p to the head h_o remaining at this time is:

$$-\frac{K_2 D_2 m}{p h_0}\left(\frac{\partial^2 h}{\partial x^2}\right) = \frac{+H m\, 8 K_2 D_2}{p W^2 H}\,\frac{0.4618}{0.9103}$$

or nearly

$$\frac{h_p}{h_0} = \frac{4 m K_2 D_2}{p W^2} \quad \ldots\ldots (58)$$

This is the criterion sought. If its value is small compared to unity or near unity then the two aquifers act together essentially as one, but if it is large compared to unity then the upper bed acts essentially alone. Intermediate values indicate a joint action in which the lower aquifer acts with reduced effectiveness due to the flow restrictions imposed by the perching layer. In a later paragraph on electric analogs an experimental procedure will be described for obtaining solutions for such cases.

This criterion was derived with reference to a point under the middle of the recharged strip and for a certain phase of the spreading. While the form of the criterion should be proper it will be wise to test it against some simple cases in order to identify the appropriate range of application more closely. Suppose, for example, the following relations exist

$$m = \frac{1}{4} D_2 \qquad p = K_2 \qquad W = D_2$$

then

$$\frac{h_p}{h_o} = \frac{D_2 K_2 D_2}{K_2 D_2^2} = 1 .$$

The relation between p and K_2 used here would make the perching layer simply a part of the lower aquifer. In this case the two aquifers should act together essentially as one but this test brings out the point that resistance to vertical flow was neglected in deriving formula 3. It seems justified to assume that the beds would act together essentially as one if the criterion is unity or even somewhat above.

Another test may be made by assuming that the upper bed acts essentially alone and then estimating the effect of leakage through the perching layer. If h_1 and h_2 represent the head changes imposed on the perching layer in the upper and lower aquifers respectively, considered as departures from an initially hydrostatic condition then the continuity condition for the upper bed would be:

$$K_1 D_1 \frac{\partial^2 h_1}{\partial x^2} - V_1 \frac{\partial h_1}{\partial t} = \frac{p(h_1 - h_2)}{m} \qquad \text{. . . . . . (59)}$$

In making this estimate it may be recognized that for a recharge case, h_2 will always be positive and that since the lower aquifer is confined its voids ratio V_2 will have a low value comparable to those of artesian aquifers, the α_2 value will be high and dissipation of pressure changes will be rapid in it. It will therefore be reasonable, for the present purposes, to neglect the factor h_2 in the above expression and assume that the leakage through the perching layer is driven by the head h_1. If this is done a modification of the method of Picard may be used to make the desired estimate. Following this procedure the right hand member will be set equal to zero and a first approximation obtained by solving the simplified differential equation, subject to the appropriate initial and boundary conditions. For a recharge of depth H over a width W at time zero the appropriate solution is given by equation (3). It may be noted that what has been done is essentially to assume that the upper bed acts alone. The discarded term is now computed from the first approximation and thrown into the right hand member as a known function. It is $\frac{p}{m} h_1$. The differential equation is now to be resolved subject to the initial and boundary conditions. This should yield a second approximation. The required relation is:

$$h_s = h_1 - \frac{p}{m} h_1 t \qquad \ldots\ldots \quad (10)$$

Where h_1 represents the solution of equation (3). An example will illustrate the use of this approximation.

Suppose the conditions are:

K_1 = .002 ft/sec D_1 = 100 feet V_1 = 0.15

α_1 = 1.333 ft^2/sec

K_2 = .002 ft/sec D_2 = 200 feet

p = .000002 ft/sec m = 20 feet $\frac{p}{m} = 10^{-7}$

$W = 2D_2$ = 400 feet   H = 10 feet

Then the criterion is:

$$\frac{4m\,K_2\,D_2}{p\,W^2} = \frac{(4)(20)(002)(200)}{(000002)(400)(400)} = 100.$$

The computation can be made as follows for the point $x = 0$.

Time days	Time seconds	$\frac{W}{2\sqrt{4\alpha t}}$	h_1	$-\frac{p}{m} h_1 t$
0	0	∞	10.00	0.000
1	86400	.295	3.23	-0.028
2	172800	.208	2.31	-0.040
3	259200	.170	1.90	-0.049
4	345600	.147	1.65	-0.057
5	432000	.132	1.48	-0.064
6	518000	.120	1.35	-0.070
7	604000	.111	1.25	-0.075
8	691000	.104	1.17	-0.081
9	777000	.098	1.10	-0.085
10	864000	.093	1.05	-0.091
11	950000	.089	1.00	-0.095
12	1036000	.085	0.96	-0.099
13	1122000	.082	0.92	-0.103
14	1208000	.079	0.89	-0.107

This computation indicates that when the criterion reaches 100 the upper bed acts essentially alone since the correction value is small when compared to the first approximation.

If the value of p had been ten times as large or $p = .00002$ ft/... the criterion value would have been 10 and the figures in the right hand column would be multiplied by 10. Then h_s by formula (60) would have reached zero at about the 12th day. It will be of interest to compare this result with the results of a computation based upon the assumption that the two beds act together. If the two beds act together then $\alpha = 4.0$. The computation is made for the point $x = 0$, as before, but using the new value of α.

Time days	$\frac{W}{2\sqrt{4\alpha t}}$	h_1	h_s
0	∞	10.00	10.00
1	0.170	1.90	2.95
2	0.120	1.35	0.95
3	0.098	1.10	0.61
4	0.085	0.96	0.39
5	0.076	0.86	0.22
6	0.069	0.78	0.08
7	0.064	0.72	0.00
8	0.060	0.68	
9	0.057	0.64	
10	0.054	0.61	
11	0.051	0.58	
12	0.049	0.55	
13	0.047	0.53	
14	0.045	0.51	

This comparison indicates that the beds may act reasonably well together until the criterion value reaches ten. The second approximation of formula (60) overestimates the drainage from the upper bed since the back pressure from the lower bed was neglected to obtain it.

The case of a square recharge area of length W on a side, may be treated with the help of the product law. Then

$$h = H f_1 f_2 \qquad \ldots\ldots (61)$$

Where f_1 and f_2 are functions of x and y of the form of equation (3) but having both sides divided by H . With this product

$$\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = H\left(f_2 \frac{\partial^2 f_1}{\partial x^2} + f_1 \frac{\partial^2 f_2}{\partial y^2}\right).$$

The flow through a unit of area of the perching layer is:

$$K_2 D_2\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = K_2 D_2 H\left(f_2 \frac{\partial^2 f_1}{\partial x^2} + f_1 \frac{\partial^2 f_2}{\partial y^2}\right)$$

The ratio of the head required to drive this flow through the perching layer to the head available at x = 0, y = 0, is as before

$$\frac{K_2 D_2 m}{p} \; \frac{H \quad 2H(.9103)(0.4618)(2)(8)}{H^2(.9103)^2 \qquad W^2} = \frac{16 K_2 D_2 m}{p W^2}$$

Then the criterion for this case is:

$$\frac{h_p}{h_o} = \frac{16 m K_2 D_2}{p W^2} \qquad \ldots\ldots (62)$$

The approximations and assumptions inherent in this method of approach would vanish if an exact solution of the equations for the case of two aquifers with an intermediate perching layer were found. The mathematical task of obtaining such a solution is a formidable one but its equivalent can be obtained by electric analog methods. If an adequate treatment were available the need for a criterion would vanish also.

Electric analogs

Sometimes when a problem becomes excessively complex so that analytical treatment is difficult and when experimentation in the original field is also difficult, the problem can be transferred to another analogous field, where experimentation is relatively simple, and the required answers found by experimentation in the analogous field. The electrical field offers great advantages in such cases because of the ease of experimentation and the speed with which electric analogs operate.

The case of the layered soil, as treated in a previous paragraph, will provide an illustration. Because of the difficulty of finding an analytical solution for the layered soil case, the effort, in this case, was directed to finding a criterion for determining when the upper bed may be considered to act alone or both beds may be considered to act together because analytical treatments are available for these simpler cases. The treatments obtained in this way are approximations. It was pointed out that the uncertainties and the need for the criterion would vanish if a solution could be found for the actual case of two aquifers separated by a semipermeable bed.

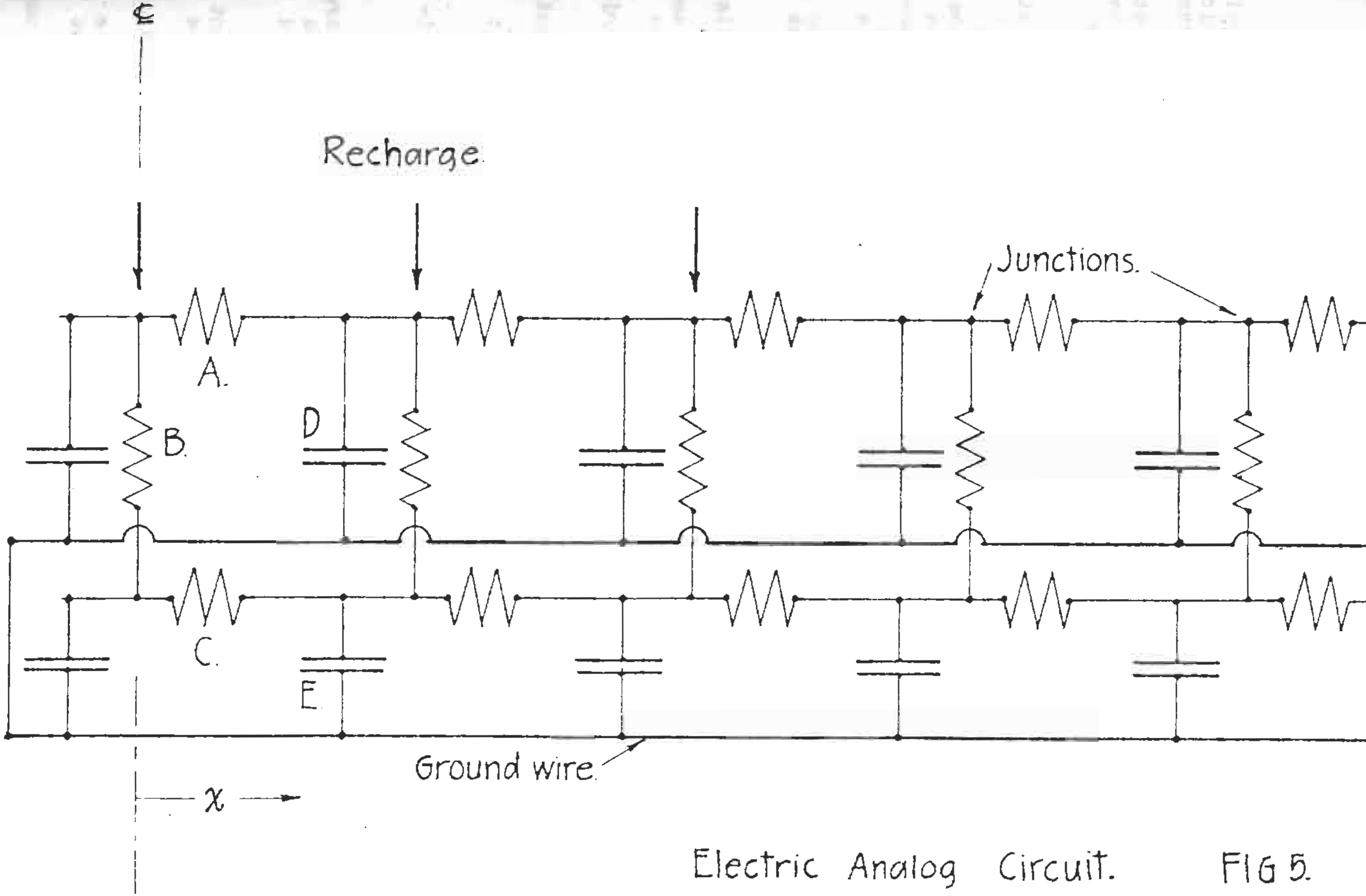
This solution can be found with the aid of an electric analog. The principles of operation may be explained with reference to figure 4. The analogous quantities are the following:

Table 1 - Analogous quantities for an electrical analog representing the flow of ground water.

Hydraulic field	Electrical field
Rate of flow	Amperes
Quantity of flow	Coulombs
Head	Voltage
Storage	Capacitance
Transmissibility	Conductance

The basic circuitry for the layered soil analog for a one dimensional case is shown in figure 5. Because the pattern is symmetrical about the origin only one half needs to be represented. The distance between junctions represents a distance on the ground in the hydraulic prototype. The resistors A, B and C represent respectively the horizontal transmissibility of the upper aquifer, the vertical transmissibility of the perching layer and the horizontal transmissibility of the lower aquifer. The condensers D and E represent the storage capacity in the distance corresponding to adjacent junctions. The method of designing the circuit is described in detail in a published paper*.

* Electrical Analogies and Electronic Computers, A Symposium Paper 2569-ASCE., Application to an Hydraulic Problem, by R.E. Glover, D.J. Hebert and C.R. Daum. 1953.



Electric Analog Circuit. FIG 5.

The procedure may be described as follows:

(a) Write the equations for the hydraulic conditions
(b) Write the electrical equations for the analog
(c) Write a system of correlation equations which when substituted in the hydraulic equations will yield the electrical equations. These fix the time scale and the size of the electrical components.

The equations for both the hydraulic and electrical systems are first written in terms of distributed quantities and the decision as to the prototype length to be represented by an interval between junctions is made later.

It will be found difficult to design these circuits to operate at rates slow enough to permit visual recording. A device originated by C.R. Daum is to use a relay of the type which makes contacts at the steady rate of about 20 per second. When the contact is made the analog runs through its solution and when this contact is opened the analog is discharged. With this arrangement the analog makes a solution 20 times a second. The output can be fed into an oscilloscope with a synchronized sweep circuit. The output will then appear as a steady curve since the persistence of the phosphor will hold the image over the 1/20 second intervals between solutions. This trace is also easily photographed. It represents a variation of head with time at the selected point.

Another reading device which works well is an oscillograph. This can be hooked up to read a number of junctions simultaneously. The rectangular coordinate plots obtained in this way will be synchronized with respect to time and a built in timer will put time marks on the record. A common paper width is 10 inches. If the paper speed is set to record the complete run on about 6 inches of paper, the record obtained can be cut to 8 by 10 inch size and bound up with the explanation typed on 8 by 10.5 inch paper. Reports of experiments can be rapidly completed in this way since it is not difficult to get 1 traces on a single sheet and the data do not have to be reworked. Some care will have to be used to keep the analog speed down to what these instruments will follow. A common undamped galvanometer frequency is 2500 cycles per second. Such an element will follow faithfully variations up to 600 cycles per second. Electronic amplification between the analog and the oscillograph will generally be needed.

When the ground water flows in two directions the analog will have a checker board pattern with each checker representing a square area of the prototype. Four resistors will come to each junction instead of the two as in the analog of figure 5 representing flow in one dimension. A radially symmetrical case can be studied by arranging the sizes of the electrical components between junctions to represent the radially expanding properties of a sector. Analogs of this type give solutions for transient cases. A skilled electronics man is needed if the costs are to be kept down. Special adaptations would have to be made if the changes of transmissibility with changes of height of the ground water mound are to be accounted for. This type of analog is adapted to conditions in which a water table is present well above the perching layer before recharge begins but it will provide solutions for any complexity of layering when these conditions are present.

Hydraulic models

The transient case presented by a perching layer above the water table, which receives recharge from above, is very difficult to treat analytically because the differential equations expressing the relationships involved are of a non-linear type. It will be difficult to treat this case by an electronic analog also unless some skilled electronics man can devise a circuit with conductances which increase with an increase in voltage. An hydraulic model can, however, provide a solution. Field conditions are simply reproduced to model scale in the laboratory and the tests are run on the model.

,It is now possible to buy glass beads of specified uniform size. These will be of help for solving the problems caused by capillarity which are troublesome in small scale model installations. Certain types of plastic sheet have properties which adapt them to model construction. They are transparent, can be sawed, drilled and machined and can be joined together securely with special cements. Calibration costs for such things as finding by trial, the size and spacing of holes in a plastic strip to properly represent the transmissibility of a perching layer are apt to be high. Much time may also be needed for these calibrations. Special tools, space, water supply and drains will also be required.

Robert G. Clark
1960

Groundwater due to recharge from a long narrow strip.- Water table present.

The line source idealization will be used to represent the mound due to the recharge. This idealization is appropriate if the recharge reaches an established water table and the height of the mound remains small when compared to the original saturated depth within the aquifer.

It will be advantageous to write formula 19 in the form:

$$\frac{h}{\left(\frac{2q_1 t}{\pi x V}\right)} = \left(\frac{x^2}{4\alpha t}\right)\sqrt{\pi}\int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u^2}\,du \quad \ldots . (6)$$

A plot of the expression in the right hand member as a function of the parameter $\frac{x}{\sqrt{4\alpha t}}$ is shown on figure 6. The use of this chart may be illustrated by means of the following example.

Example

The bed of a previously dry watercourse is used as a recharge area. When supplied with recharge water the inundated width averages 40 feet. Water is absorbed at the rate of 0.5 foot per day over the inundated area. An observation well 100 feet away from the center of the recharged strip shows a rise of 2.17 feet as a result of continuous recharge over a period of 30 days. It is known that the water table was stable at a depth of 55 feet in the observation well before the recharge operation began. It is desired to determine the aquifer constant α from these data.

Solution:

h = 2.17 feet

t = 2592000 seconds (30 days)

$q_1 = \frac{(40)(0.5)}{86400} = .0002316$ ft²/sec.

V = 0.12 (from a supplementary experiment)

x = 100 feet

$$\left(\frac{2 q_1 t}{\pi x V}\right) = \frac{(2)(.0002316)(2592000)}{(3.1416)(100)(0.12)} = 31.84$$

$$\frac{h}{\left(\frac{2 q_1 t}{\pi x V}\right)} = 0.068$$

$$\left(\frac{x}{\sqrt{4\alpha t}}\right) = .042 \quad \text{and} \quad \left(\frac{x^2}{4\alpha t}\right) = .00177$$

then

$$\alpha = \frac{x^2}{4t\ .00177} = \frac{10000}{(10,368,000)(.00177)} = 0.54\ \text{ft}^2/\text{sec.}$$

Since the chart of figure 6 can yield two values for $(x/\sqrt{4\alpha t})$ for a given value of $h/\left(\frac{2q_1 t}{\pi xV}\right)$ it is much better to use a series of values of h read from an observation well to make the determination of α. A graphical procedure provides an effectiv means to do this. Suppose the readings in the first two columns of the following table are available for the well at $x = 100$ feet.

time	h (feet)	time (seconds)	$\frac{h}{\left(\frac{2q_1 t}{\pi xV}\right)}$	$\frac{x}{\sqrt{t}}$
1 hr	0.05	3600	1.10	1.67
5 hr	0.07	18000	0.31	0.745
10 hr	0.13	36000	0.31	0.527
1 day	0.28	86400	0.27	0.340
2 "	0.43	172300	0.22	0.241
5 "	0.82	423000	0.155	0.152
10 "	1.20	864000	0.113	0.108
20 "	1.74	1728000	0.082	0.0760
30 "	2.16	2592000	0.068	0.0621
60 "	3.18	5184000	0.050	0.0440

The derived quantities in the last two columns are now plotted on a logarithmic chart as shown in figure 7 and this chart is superimposed on the chart of figure 6 in such a way as to make the observations of figure 7 fit the type curve of figure 6 as well as possible. An index point is chosen on figure 6. In this case it represents the point 1, 1. When the adjustment has been made the point where the index falls on figure 7 is marked. This point is indicated by a cross on figure 7. The horizontal axes coincide on the two charts because the same ordinate is used on each but there is a shift along the horizontal axis sufficient to bring the index of figure 7 to the point $(x/\sqrt{t}) = 1.42$ on figure 7. The abscissa quantities are different on the two charts but the scales are logarithmic and the above shift represents the factor needed to make them the same.
Then, from figure 6

$$\frac{x}{\sqrt{4\alpha t}} = 1.00$$

and from figure 7

$$\frac{x}{\sqrt{t}} = 1.42$$

By substitution

$$\frac{1}{\sqrt{4\alpha}} = \frac{1.00}{1.42}$$

or

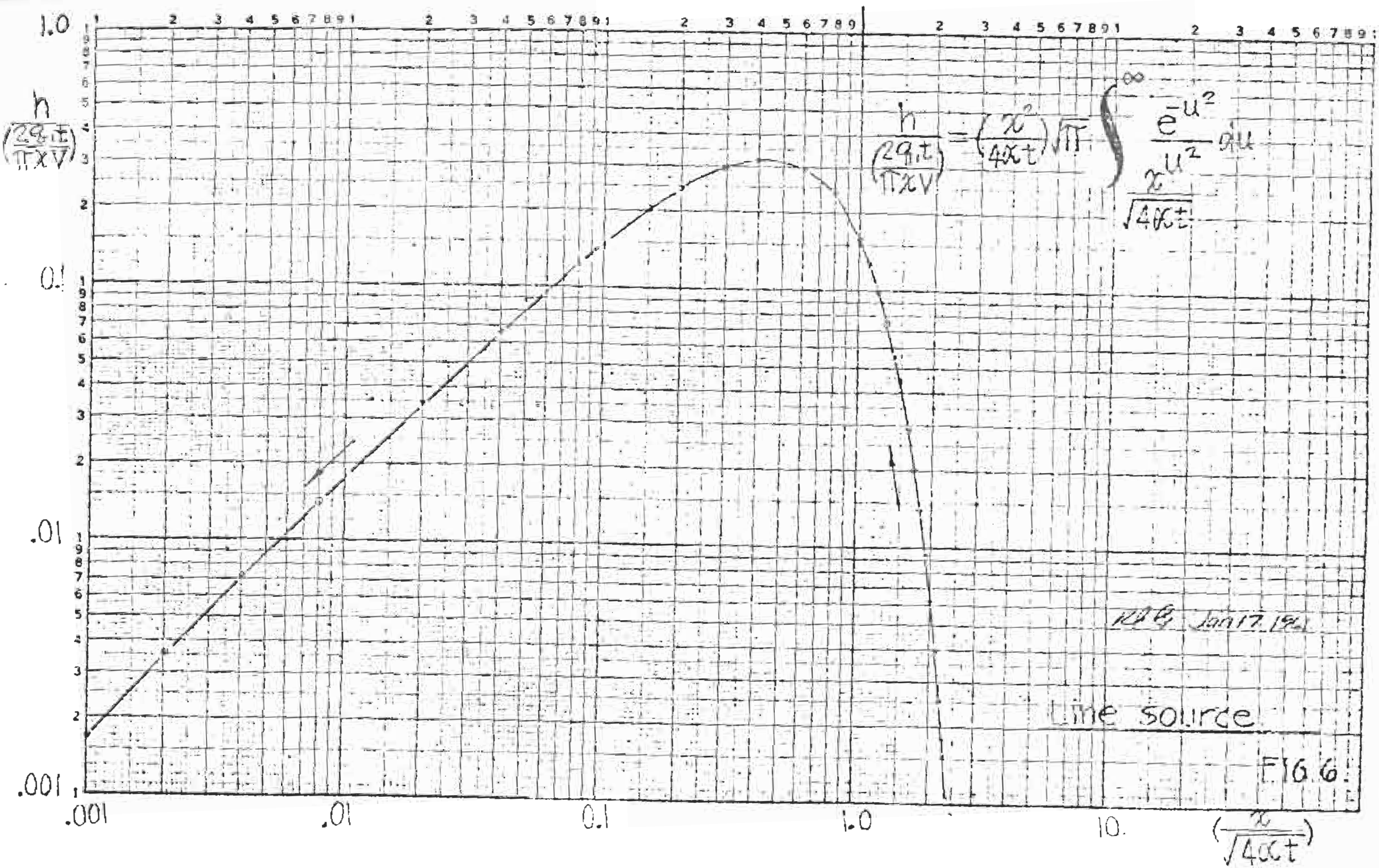
$$4\alpha = 1.42^2$$

then

$$\alpha = \frac{1.42^2}{4} = 0.51$$

This is the quantity sought. The use of the series of readings removes the ambiguity that is present when only one reading is used. The two values of α differ slightly due to variations in reading the charts and to small inconsistencies in the values representing the observation well readings.

Jan 17 1961


1.0
0.1
.01
.001
$\frac{h}{\left(\frac{2q_t}{\pi x V}\right)}$
$\frac{h}{\left(\frac{2q_t}{\pi x V}\right)} = \left(\frac{x^2}{4\alpha t}\right)\sqrt{\pi}\int_{\frac{x}{\sqrt{4\alpha t}}}^{\infty}\frac{e^{-u^2}}{u^2}\,du$
RDG Jan 17 1981
Line source
FIG 6.
.001
.01
0.1
1.0
10.
$\left(\frac{x}{\sqrt{4\alpha t}}\right)$

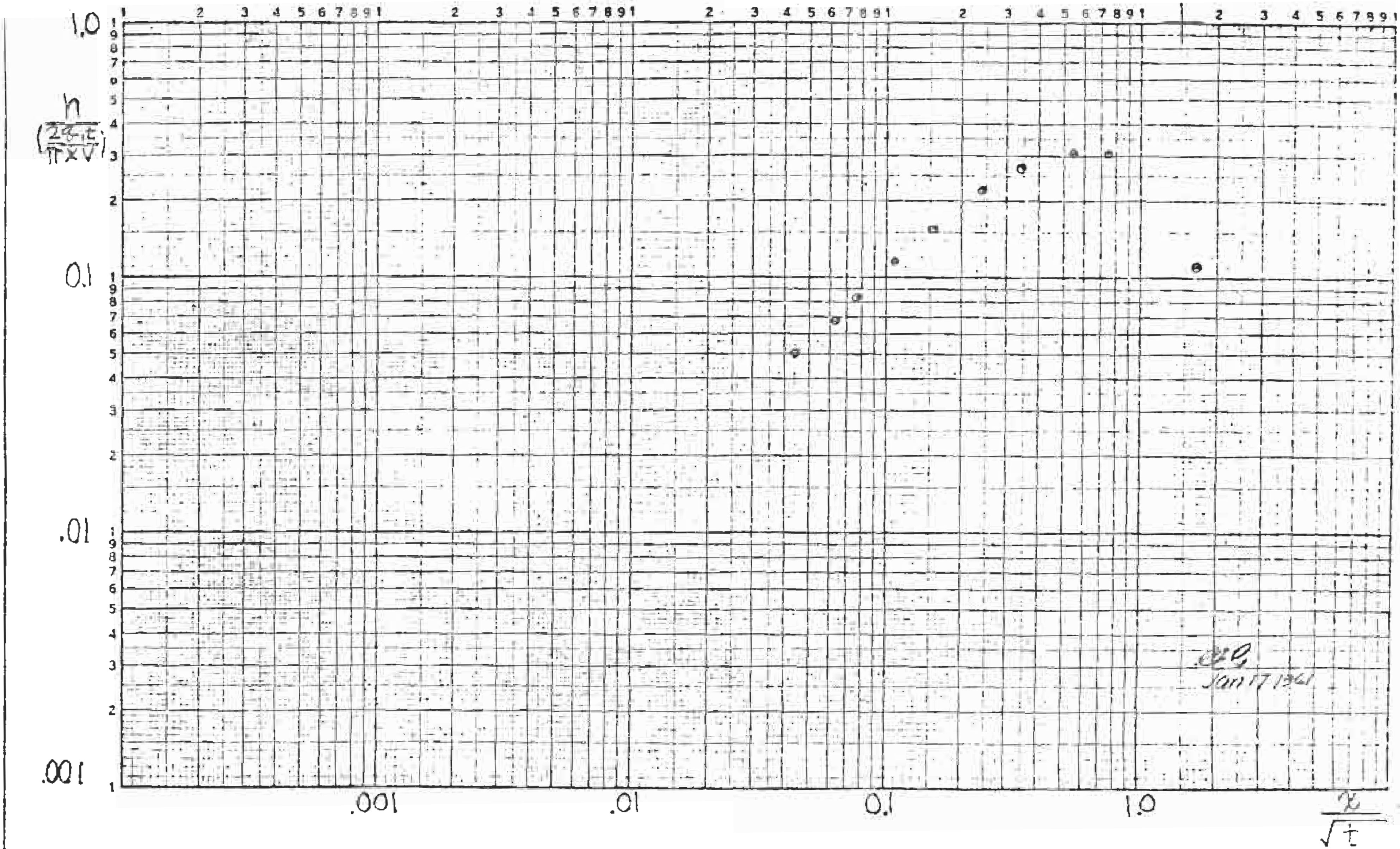


FIG. 7.

Water table below the perching layer.- Transient case.

The configuration treated in this case is illustrated on figure 2. A bed of permeability K overlies a perching layer of thickness m and permeability p. A capillary tension c acts at the bottom of the perching layer. A continuous recharge at the rate q_1 builds up a mound of height h at the point x and the time t. The flow through the saturated depth h at the point x is:

$$F = Kh \frac{\partial h}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

and the condition of continuity is:

$$\frac{\partial F}{\partial x} = \frac{p}{m} h + p\frac{(m + c)}{m} + V \frac{\partial h}{\partial t} \quad \ldots\ldots\ldots (64)$$

where V represents the ratio of the fillable voids in the aquifer to the whole volume and t represents time. Elimination of F between these two expressions would yield a non-linear partial differential equation whose solution should represent the transient characteristics of the mound. Because of the difficulty of solving the differential equation the shape of the mound will be found by using an iteration procedure. A first approximation will first be obtained, based upon the assumption that the mound is triangular in shape. Quantities obtained from this first approximation can be used in the right hand member of equation (64) and values of F can then be computed by approximate integration methods. If these values are used in equation (25) a second integration can be made to yield h. These h values represent a second approximation. A third approximation can be based upon the second approximation h values. The true solution is obtained when two successive approximations become identical.

The first approximation

If, as a first approximation, the mound is considered to be triangular in shape with a height at the apex of h_a and a base width of b, on either side of the apex, then the volume of water in the left half of the mound will be

$$v_1 = \frac{b\, h_a\, V}{2} \quad \ldots\ldots (65)$$

The flow into the left half must be $q/2$ then

$$\frac{K\, h_a^2}{b} = \frac{q_1}{2} \quad \ldots\ldots (66)$$

It can be written

$$\frac{q_1}{2} = \frac{pb(m + c)}{m} + \frac{pb\,h_a}{2m} + \frac{\partial v_1}{\partial t} \qquad \ldots\ldots (67)$$

By substitution from equations (65) and (66) this expression can be put into the form:

$$h_a^2 \frac{dh_a}{dt} = \frac{q^2}{6KV} - 2\,\frac{p(m + c)h_a^2}{3mV} - \frac{ph_a^3}{3mV} \qquad \ldots\ldots (68)$$

The variables in this equation are separable and this permits an integration in the form:

$$\int_0^h \frac{h_a^2\, dh_a}{\dfrac{q^2}{6KV} - \dfrac{2p(m + c)h_a^2}{3mV} - \dfrac{ph_a^3}{3mV}} = t \qquad \ldots\ldots (69)$$

This expression can be evaluated for any given case by Simpson's rule or graphical integration. The corresponding values of b can then be obtained from equation (66).

<u>The second approximation</u>

At this point it will be desirable to introduce some dimensionless parameters selected to generalize the results obtained and to simplify the work of computation. These are:

$$\psi = \frac{F}{q1}\,. \quad \eta = \frac{h}{(m + c)}\,. \quad \xi = \frac{(x_o - x)p(m + c)}{mq}\,. \quad \theta = \frac{p}{mV}t\,.$$

It is found that each case is identified by the parameter:

$$\zeta = \frac{mq^2}{pK(m + c)^3} \qquad \ldots\ldots\ldots\ldots (71)$$

In this new terminology equations (69) and (66) take the forms

$$6 \int_{0}^{\eta_a} \frac{\eta_a^{2}\, d\eta_a}{\zeta - 4\eta_a^{2} - 2\eta_a^{3}} = \theta \qquad \ldots\ldots\ (72)$$

and

$$\zeta_a = \frac{2}{\zeta}\, \eta_a^{2} \qquad \ldots\ldots\ (73)$$

The basic relationships of formulas (25) and (64) become

$$\psi = -\frac{1}{\zeta}\, \eta \frac{\partial \eta}{\partial \xi} \qquad \ldots\ldots\ (74)$$

and

$$-\frac{\partial \psi}{\partial \xi} = \eta + 1 + \frac{\partial \eta}{\partial \theta} \qquad \ldots\ldots\ (75)$$

(The minus signs appear here because ξ is referred to an origin under the apex of the mound, whereas in the treatment of of the steady state case x was measured from the toe of the mound.)

A first approximation chart for the case $\zeta = .025$ is shown on figure 8. The apex values of η_a were obtained by Simpson's rule integration from equation (72) and the corresponding ξ_a values were obtained from equation (73). In the first approximation chart the corresponding η_a and ξ_a points are joined by straight lines. This conforms to the first approximation assumption that the mound is triangular in shape.

A series of curves of η versus θ obtained from this first approximation are shown by the solid lines of figure 9 and some curves of $\frac{\partial \eta}{\partial \theta}$ versus ξ obtained from them are shown on figure 10. Values can now be assigned to two quantities in the righthand member of equation (75). The integrations required to find ψ and η were made by use of Simpson's rule. A sample computation is shown in the table below.

Sample computation for θ = .010 based upon the first approximation case for ζ = .025.

ξ	η	$\frac{\partial \eta}{\partial \theta}$	$(\eta + 1 + \frac{\partial \eta}{\partial \theta})$	ψ	$-\int \psi \, d\xi$	η
.00	.046	1.35	2.40	0.500	.0460	.0479
.01	.044	1.40	2.44			
.02	.041	1.46	2.50	0.451		
.03	.038	1.52	2.56			
.04	.035	1.59	2.63	0.400	.0279	.0373
.05	.033	1.63	2.66			
.06	.030	1.69	2.72	0.347		
.07	.028	1.75	2.78			
.08	.025	1.81	2.83	0.291	.0141	.0265
.09	.021	1.87	2.89			
.10	.019	1.93	2.95	0.233		
.11	.017	1.98	3.00			
.12	.014	2.04	3.05	0.173	.0048	.0154
.13	.011	2.10	3.11			
.14	.006	2.16	3.17	0.110		
.15	.003	2.21	3.21			
.16	.000	2.27	3.27	0.046	.0003	.0041
.1745		$= \frac{.046}{3.27}$	= .0145	0.000		

The integration for ψ is made in the direction of $+\xi$ since, at $\xi = 0.$, ψ = 0.5000 in all cases. The integration of ψ with respect to ξ is made in the direction of $-\xi$ since the boundary condition $\eta = 0$ is known at the toe of the mound. The η values are obtained from the integral of equation (74).

$$\eta^2 = 2\zeta \int \psi \, d\xi \qquad \text{. . . . . (76)}$$

The new values so obtained are plotted on figure 9, a graphical interpolation is used to produce the dotted curves and from these, values are obtained to plot the second approximation plot of figure 11. This could be used as a starting point for a third approximation. In this case, however, the change from the first approximation was small and the third approximation should not differ significantly from the second approximation. The ultimate accuracy attainable by the process described is limited by the accuracy of determination of slopes from the curves of figure 9. The scatter of the plotted points of figure 10 is due to this difficulty. If greater precision should be found desirable it could probably be attained by rearranging the computations to use tables prepared for computing the derivatives from ordinates of a curve in the neighborhood of a point.

[illegible]
Jan 17 1961.

Spreading of a mound due to a line source

First approximation $\varsigma = 0.025$ RGS 1-12-61

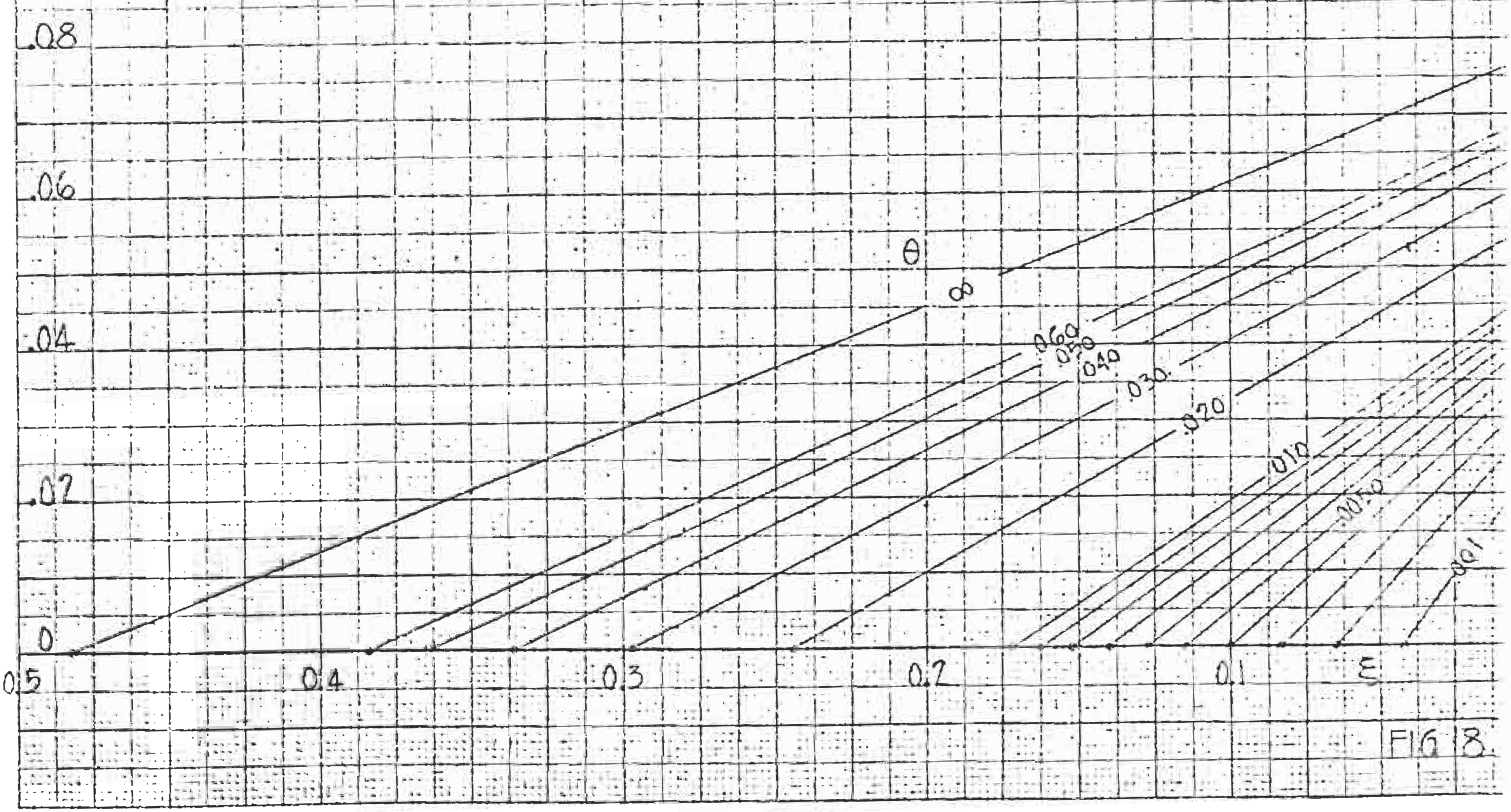
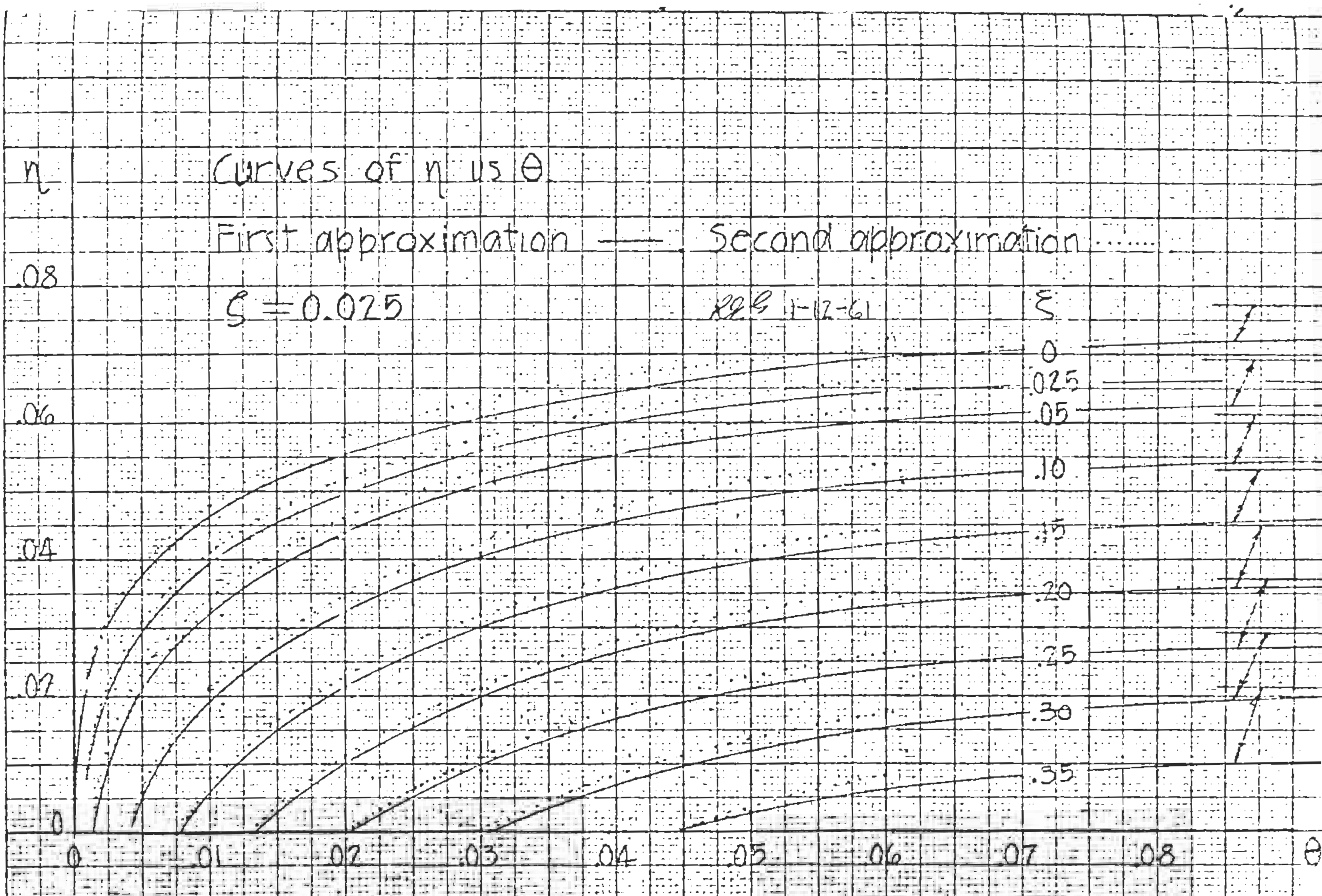
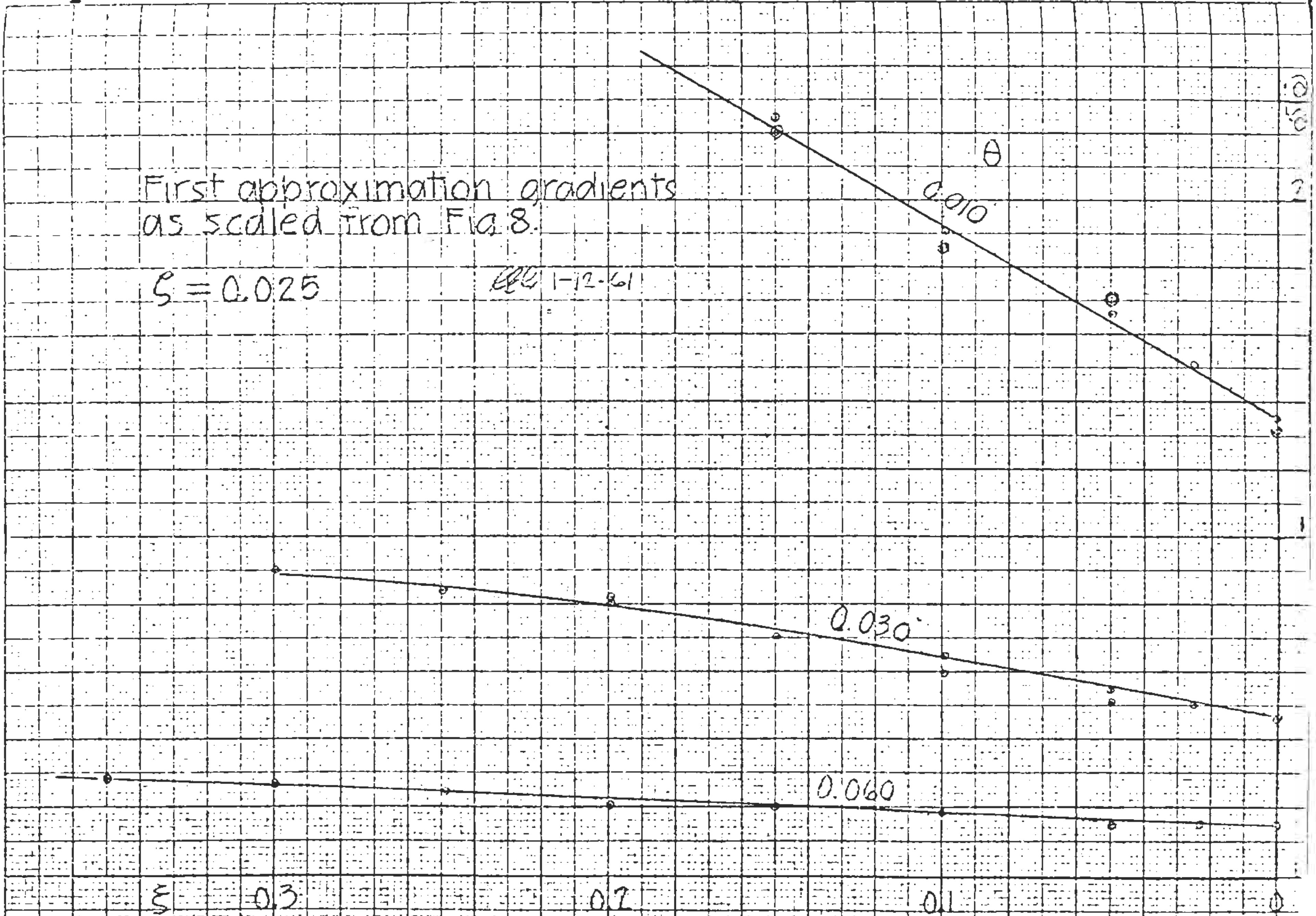


FIG 8



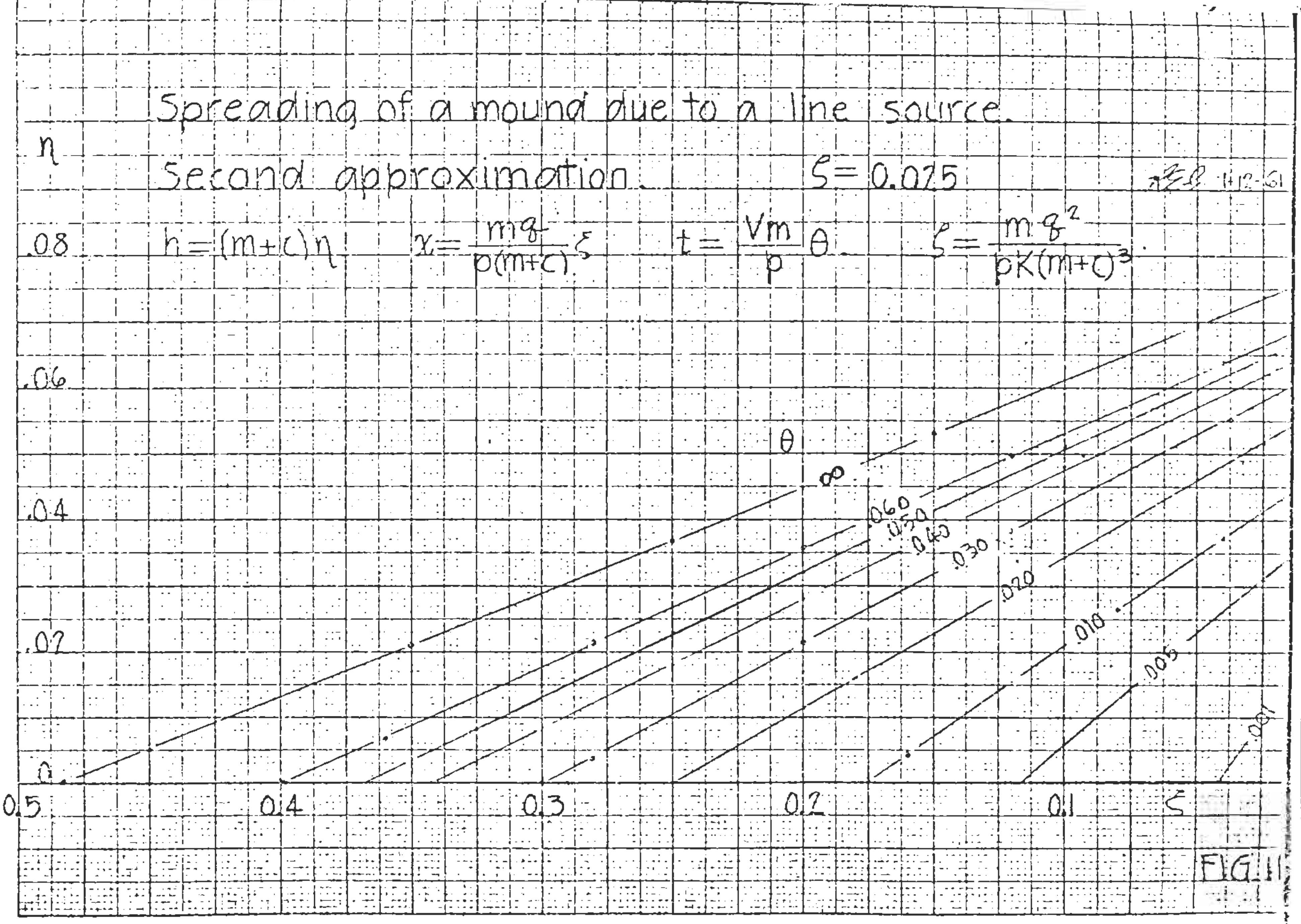
η
Curves of η vs θ
First approximation —— Second approximation ......
ς = 0.025
1-12-61
ξ
0
.025
.05
.10
.15
.20
.25
.30
.35
.08
.06
.04
.02
0
0
.01
.02
.03
.04
.05
.06
.07
.08
θ

FIG. 9.



First approximation gradients
as scaled from Fig 8.
ς = 0.025
1-12-61
θ
0.010
0.030
0.060
ξ
0.3
0.2
0.1
0
2
1

FIG 10



Spreading of a mound due to a line source.
Second approximation.
$\zeta = 0.075$
$h = (m+c)\eta$
$x = \frac{mq}{p(m+c)}\xi$
$t = \frac{Vm}{p}\theta$
$\zeta = \frac{mq^2}{pK(m+c)^3}$
η
.08
.06
.04
.02
0
0.5
0.4
0.3
0.2
0.1
ξ
θ
∞
.060
.050
.040
.030
.020
.010
.005
.001

FIG. 11

Spreading of a ground water mound due to a continuous recharge from a long strip of width W. Water table present.

The case of an instantaneous recharge is given by formula 3. The solution for the case of continuous recharge can be derived from this formula by a process of integration. If $dH = R\,d\eta$ represents the depth of an increment of ground water applied over the width W at the time η due to a constant recharge rate R operating over the time interval $d\eta$ then the height of the ground water mound at the point x at the time t will be given by the expression

$$h = R\int_0^t\left(\frac{1}{\sqrt{\pi}}\int_{u_1}^{u_2} e^{-u^2}\,du\right)d\eta \qquad \ldots\ldots (77)$$

where, in this case

$$u_1 = \frac{\left(x - \frac{W}{2}\right)}{\sqrt{4\alpha(t-\eta)}} \qquad u_2 = \frac{\left(x + \frac{W}{2}\right)}{\sqrt{4\alpha(t-\eta)}} \qquad \ldots\ldots (78)$$

Let $\xi = (t - \eta)$ then $d\xi = -d\eta$ and the above integral becomes, when expressed in terms of the probability integral:

$$h = \frac{R}{2}\int_0^t\left(\frac{2}{\sqrt{\pi}}\int_{u_1}^{u_2} e^{-u^2}\,du\right)d\xi \qquad \ldots\ldots (79)$$

This expression can be put in the form:

$$h = \frac{R}{2}\int_0^t\left(\frac{2}{\sqrt{\pi}}\int_0^{u_2} e^{-u^2}\,du\right)d\xi - \frac{R}{2}\int_0^t\left(\frac{2}{\sqrt{\pi}}\int_0^{u_1} e^{-u^2}\,du\right)d\xi \qquad \ldots\ldots (80)$$

an integration by parts with

$$w = \frac{e}{\sqrt{\pi}} \int_0^{u_2} e^{-u^2} du \qquad dv = d\xi \qquad \ldots\ldots (31)'$$

permits the evaluation of the first of these integrals in the form

$$\int_0^t \left(\frac{2}{\sqrt{\pi}} \int_0^{u_2} e^{-u^2} du\right) d\xi = t\left[\frac{2}{\sqrt{\pi}} \int_0^{u_4} e^{-u^2} du + \frac{2u_4^2}{\pi} \sqrt{\pi} \int_{u_4}^{\infty} \frac{e^{-u^2}}{u^2} du\right] \qquad . (32)'$$

where

$$u_4 = \frac{(x + \frac{W}{2})}{\sqrt{4\alpha t}} \qquad \ldots\ldots (33)'$$

The second integral is of the same form but has the parameter u_4 of the above expression replaced by

$$u_3 = \frac{(x - \frac{W}{2})}{\sqrt{4\alpha t}} \qquad \ldots (34)'$$

The first integral in the bracket is the Probability Integral. It has been tabulated in terms of the upper limit. (See tables) The second integral has been tabulated by Mr. M.W. Bittinger. This table may be found in reference 8. A plot of the function

$$M(\beta) = \frac{2}{\sqrt{\pi}} \int_0^{\beta} e^{-u^2} du + \frac{2\beta^2}{\pi} \sqrt{\pi} \int_{\beta}^{\infty} \frac{e^{-u^2}}{u^2} du \qquad \ldots (35)'$$

is shown on figure 1?. The solution for the height h of a ground water mound under a long strip of width W, at the time t, due to a continuous recharge at the rate R can now be expressed in the form:

$$\left(\frac{h}{Rt}\right) = \frac{1}{2}\left[M(u_4) - M(u_3)\right] \qquad \ldots (36)'$$

The symbol R represents the rate at which the water table would rise under the strip if all of the recharge water were retained within the width W. A recharge at a constant rate is assumed and the quantity R will therefore be a constant. When x is less than $\frac{W}{2}$ some negative limits will appear in the formula for M.

Plot of the M function

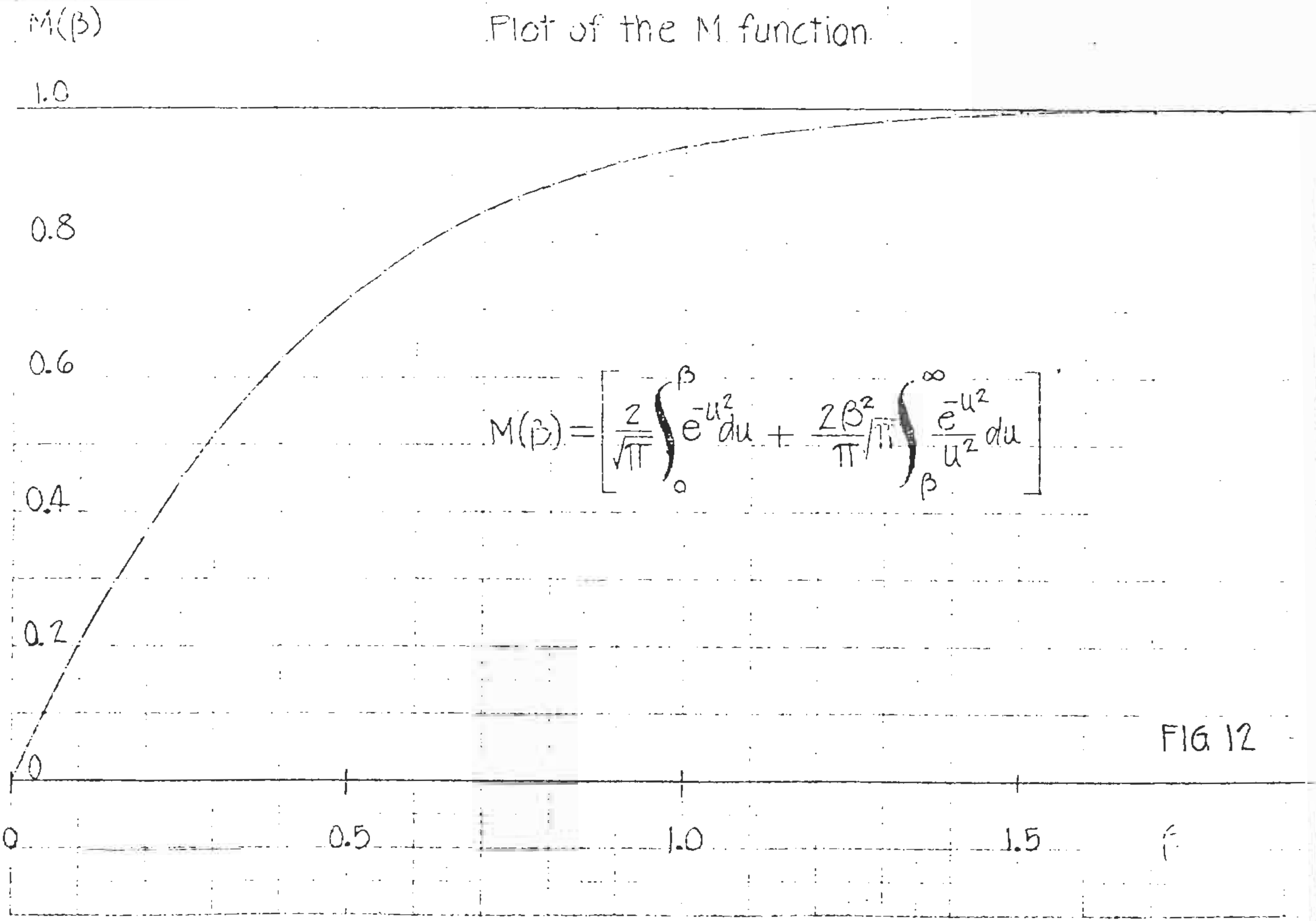


FIG 12

hese negative limits will make the values of M negative.

Example

Suppose a long recharge plot is 330 feet wide and is recharged at such a rate that a one foot depth of water sinks into its surface each day. The aquifer properties are:

$K = .00015$ ft/sec

$D = 100$ feet

$V = 0.15$ (dimensionless)

$\alpha = \dfrac{KD}{V} = 0.1 \text{ ft}^2/\text{sec}$

Estimate the rise of the water table under the middle of the strip at the end of 15 days of recharge.

Solution:

One foot per day is equivalent to $\dfrac{1}{86400}$ feet per second

then

$$R = \frac{1}{(86400)(0.15)} = .0000772 \text{ ft/sec.}$$

15 days is equivalent to $(86400)(15) = 1,296,000$ seconds

$W = 330$ feet $\quad \dfrac{W}{2} = 165$ feet

Since $x = 0$ $\quad u_4 = \dfrac{165}{\sqrt{4\alpha t}} = \dfrac{165}{\sqrt{518400}} = \dfrac{165}{720} = 0.229$

From the curve of figure 12

$M(u_4) = 0.42$

With $x = 0$ $\quad u_3 = -\dfrac{165}{\sqrt{4\alpha t}} = -0.229$ then

$M(u_3) = -0.42$ and

$$\frac{h}{Rt} = \frac{1}{2}\left[0.42 + 0.42\right] = 0.42$$

$$Rt = (.0000772)(1296000) = 100.0 \text{ feet}$$

then

$$h = (100.0)(0.42) = 42.0 \text{ feet}$$

A plot of the profile of the mound at the end of 15 days of recharge can be made as follows:

x	$(x+\frac{W}{2})$	$(x-\frac{W}{2})$	u_4	u_3	$M(u_4)$	$M(u_3)$	h (feet)
0	165	-165	.229	-.229	0.418	-0.418	41.3
100	265	- 65	.368	-.090	0.592	-0.134	38.8
165	330	0	.458	0	0.631	0	34.0
200	365	+ 35	.507	+.049	0.725	+0.100	31.2
300	465	+135	.646	+.138	0.817	+0.350	23.4
400	565	+235	.785	+.326	0.880	+0.545	16.8
500	665	+335	.924	+.465	0.925	+0.637	11.9
750	915	+585	1.271	+.812	0.930	+0.890	4.5
1000	1165	+835	1.389	+1.160	0.938	+0.967	1.0

The signs of the x values in the above table can be made negative without changing the computed values of h.

<u>Determination of aquifer constants.</u>

If aquifer constants are to be derived from observations of the behavior of the ground water mound then the chart of figure 13 is needed. This is prepared by use of formula 36. An example will illustrate its use.

<u>Example:</u>

A long recharge plot is 60 feet wide and is supplied with water sufficient to cover the recharge area to a depth of 0.72 feet each day. At the end of 10 days of recharge the mound under the center of the plot has risen 10.23 feet. It is known from tests of samples that V = .07. The aquifer constant α is to be determined.

<u>Solution:</u>

$$R = \frac{(0.72)}{(86400)(.07)} = .0001190 \text{ ft/sec}$$

There are 86400 seconds in a day and therefore

$$t = (10)(86400) = 364000 \text{ seconds.}$$

$$Rt = (864,000)(.000,1190) = 102.81 \text{ feet}$$

$$\frac{h}{Rt} = \frac{10.23}{102.81} = 0.1$$

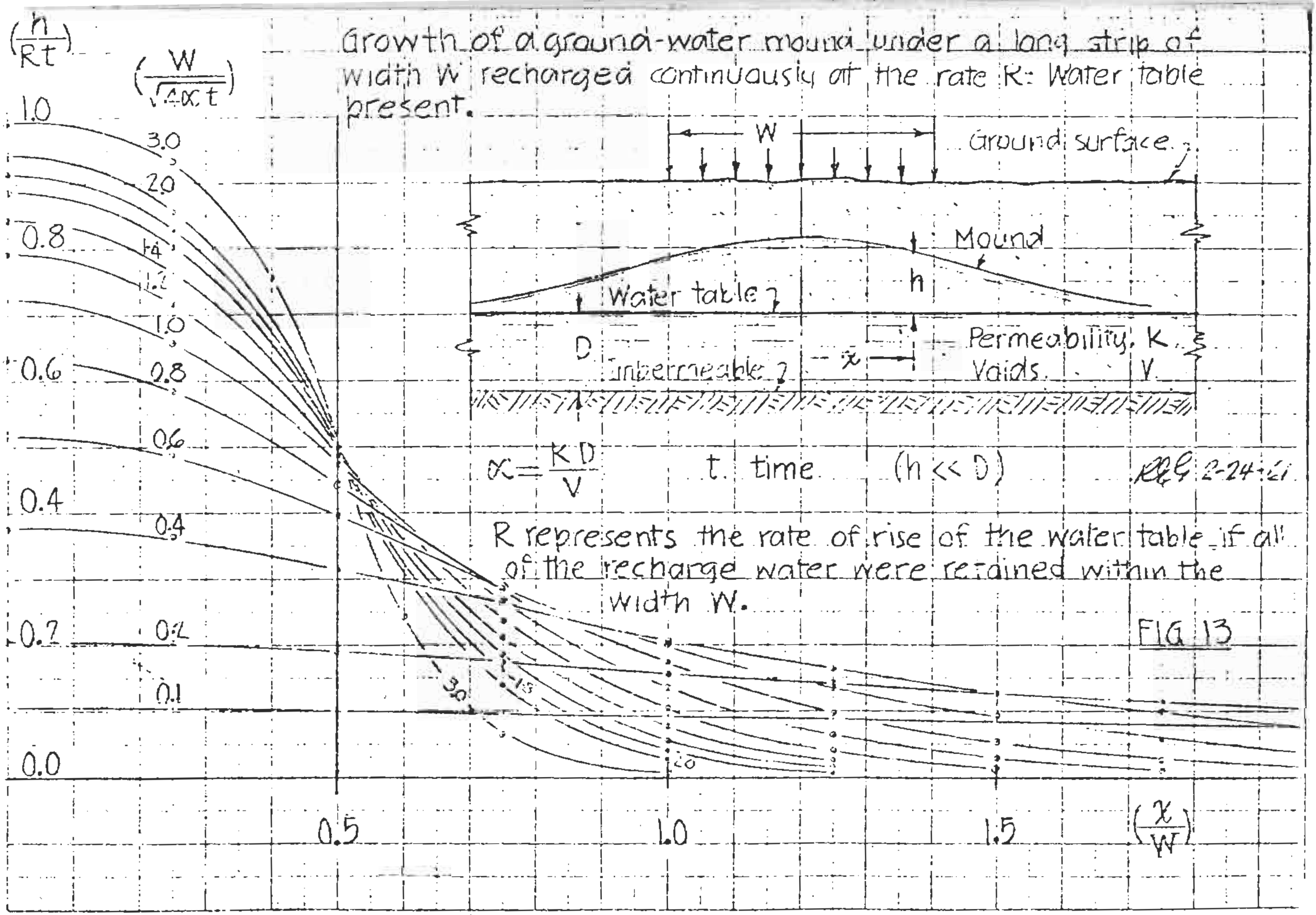

Growth of a ground-water mound under a long strip of width W recharged continuously at the rate R: Water table present.
W
Ground surface
Mound
h
Water table
D
Permeability, K
Voids, V
Impermeable
x
$\alpha = \frac{KD}{V}$
t. time
(h << D)
RRG 2-24-61
R represents the rate of rise of the water table if all of the recharge water were retained within the width W.
$\left(\frac{h}{Rt}\right)$
$\left(\frac{W}{\sqrt{4\alpha t}}\right)$
1.0
0.8
0.6
0.4
0.2
0.0
3.0
2.0
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.1
0.5
1.0
1.5
$\left(\frac{x}{W}\right)$

FIG 13

From the chart of Figure 13 for $\frac{h}{Rt} = 0.1$ and $\frac{x}{W} = 0$

read

$$\frac{W}{\sqrt{4 \alpha t}} = 0.1$$

then

$$\frac{W^2}{4 \alpha t} = .01 \qquad \alpha = \frac{W^2}{(4)(.01)t} = \frac{60^2}{(4)(.01)(864000)} = 0.1042 \ \frac{ft^2}{sec}$$

with

$V = .07$ the transmissibility is:

$KD = (0.1042)(.07) = .0073 \ ft^2/sec.$

Robert E. Glover
February 24, 1961

Rise of a ground water mound due to a continuous recharge applied over a rectangular strip of width W and length L.- Water table present.

Suppose water infiltrates the surface of a rectangular plot at the constant rate i feet per second and moves downward under the action of gravity to the water table. If the incoming water were retained within the boundaries of the recharge area it would cause the water table to rise at the rate R = (i/V) where V represents the ratio of fillable voids in the aquifer material to the total volume. Following the product law, as described on page 2, the spreading of an increment of recharge $R d\eta$ occurring during the time interval $d\eta$ may be expressed by a product of two functions of the type of formula 3.

An integration with respect to time will then yield the rise h at the time t in the form:

$$h = R \int_0^t \left(\frac{1}{\sqrt{\pi}} \int_{u_1}^{u_2} e^{-u^2} du \right) \left(\frac{1}{\sqrt{\pi}} \int_{u_3}^{u_4} e^{-u^2} du \right) d\eta \qquad (87)$$

If

$$u_1 = \frac{(x - \frac{W}{2})}{\sqrt{4\alpha(t-\eta)}} \cdot \quad u_2 = \frac{(x + \frac{W}{2})}{\sqrt{4\alpha(t-\eta)}} \quad u_3 = \frac{(y - \frac{L}{2})}{\sqrt{4\alpha(t-\eta)}} \cdot \quad u_4 = \frac{(y + \frac{L}{2})}{\sqrt{4\alpha(t-\eta)}} \qquad (88)$$

Let

$$\xi = \frac{\eta}{t} \qquad d\xi = \frac{1}{t} d\eta \qquad (89)$$

Then the above integral becomes

$$\frac{h}{Rt} = \frac{1}{4} \int_0^1 \left(\frac{2}{\sqrt{\pi}} \int_{u_1}^{u_2} e^{-u^2} du \right) \left(\frac{2}{\sqrt{\pi}} \int_{u_3}^{u_4} e^{-u^2} du \right) d\xi \qquad (90)$$

Where

$$u_1 = \frac{(x - \frac{W}{2})}{\sqrt{4\alpha t(1-\xi)}} \quad u_2 = \frac{(x + \frac{W}{2})}{\sqrt{4\alpha t(1-\xi)}} \quad u_3 = \frac{(y - \frac{L}{2})}{\sqrt{4\alpha t(1-\xi)}} \cdot \quad u_4 = \frac{(y + \frac{L}{2})}{\sqrt{4\alpha t(1-\xi)}} \qquad (91)$$

This [illegible] be evaluated by a Simpson's 1 [illegible] integration as illustrated below for the point at the middle of one side of a square recharge plot of width W where $x = W/2.$, $y = 0$. The evaluation is made for $(W/\sqrt{4 \alpha t}) = 0.6$. For this case

$$\left(x-\frac{W}{2}\right)=0. \quad \left(x+\frac{W}{2}\right)=W. \quad \left(y-\frac{W}{2}\right)=-\frac{W}{2}. \quad \left(y+\frac{W}{2}\right)=\frac{W}{2}$$

Then $u_1 = 0$ and $-u_3 = u_4$. The Probability integral for the range u_3 to u_4 is twice the Probability integral for the range 0 to u_4. This factor of 2 and the one fourth factor outside the integral are all absorbed into the divisor for the Simpson's rule integration. In this case the divisor is 60. The result of the computation is $(h/Rt) = 0.2132$

ξ	$\frac{1}{\sqrt{1-\xi}}$	u_2	$\frac{2}{\sqrt{\pi}}\int_0^{u_2} e^{-u^2} du$	u_4	$\frac{2}{\sqrt{\pi}}\int_0^{u_4} e^{-u^2} du$	Product	Simpson's factor
0.0	1.0000	.6000	.6039	.3000	.3286	.1984	1
0.1	1.0541	.6325	.6289	.3162	.3452	.2171	4
0.2	1.1181	.6709	.6573	.3354	.3647	.2397	2
0.3	1.1952	.7171	.6895	.3591	.3884	.2678	4
0.4	1.2910	.7746	.7267	.3876	.4164	.2952	2
0.5	1.4142	.8485	.7698	.4242	.4514	.3475	4
0.6	1.5810	.9486	.8202	.4743	.4976	.4081	2
0.7	1.8255	1.0955	.8466	.5481	.5559	.4706	4
0.8	2.2361	1.3417	.9422	.6705	.6570	.6190	2
0.9	3.1616	1.8970	.9925	.9483	.8201	.8139	4
1.0	∞	∞	1.0000	∞	1.0000	1.0000	1
						0.2132	

The charts of figures 14 and 15 are based upon a series of such integrations. Each integration provides one point on the chart. Figure 14 shows the rise at the center of a square recharge plot. The rise at the center of a circular recharge plot of the same area is also shown for purposes of comparison. The rise at the center of the circular recharge area was computed by use of formula 1[illegible].

Figure 15 shows successive profiles of the mound along a line passing from the center of the plot through the middle of a side. This chart may be used either to estimate the height of a mound or to determine the aquifer properties from observed data. An example of the use of these charts is given below.

Suppose

$K = .00015$ ft/sec $\qquad \alpha = \frac{KD}{V} = 0.1$ ft^2/sec

$D = 100$ feet

$V = 0.15$ (dimensionless)

Recharge is applied to a square plot 330 feet on a side at the rate of one foot per day for 15 days. Compute the height of the ground water mound at the end of this period.

$$R = \frac{1.0}{(0.15)(86400)} = 77.16 \times 10^{-6} \text{ ft/sec.}$$

Fifteen days is $(15)(86400) = 1296000$ seconds
$Rt = (77.16)(10)^{-6}\ (1296000) = 100$ feet

$$\frac{W}{\sqrt{4\alpha t}} = \frac{330}{\sqrt{(4)(0.1)(1296000)}} = \frac{330}{720} = 0.453$$

From figure 15 for $x = 0$, $y = 0$, $\frac{h}{Rt} = 0.207$

Then $h = (100)(0.207) = 20.7$ feet. This is the estimated rise at the center of the plot. The result can be compared with that shown on page 6.

Under the edge of the plot where $x = \frac{W}{2}$, $y = 0$,

$\frac{h}{Rt} = 0.152$. Then $h = (100)(0.152) = 15.2$ feet.

At $x = W$, $y = 0$, $\frac{h}{Rt} = 0.08$. $h = (100)(0.08) = 8.0$ ft.

Figure 14 will be useful where the rise of the ground water mound under the center of the plot is observed and it is desired to determine the aquifer constant α. Suppose a rise of 20.7 feet is observed under the center of the square plot described above after the plot has been recharged at the rate of 1 foot per day for 15 days. It is assumed that V is known from a laboratory measurement to be 0.15.

then

$$\frac{h}{Rt} = \frac{20.7}{100} = 0.207$$

From figure 14 for $\frac{h}{Rt} = 0.207$ read $\frac{W}{\sqrt{4\alpha t}} = 0.43$

Then

$$\sqrt{4\alpha t} = \frac{W}{0.43} = \frac{330}{0.43} = 767.4$$

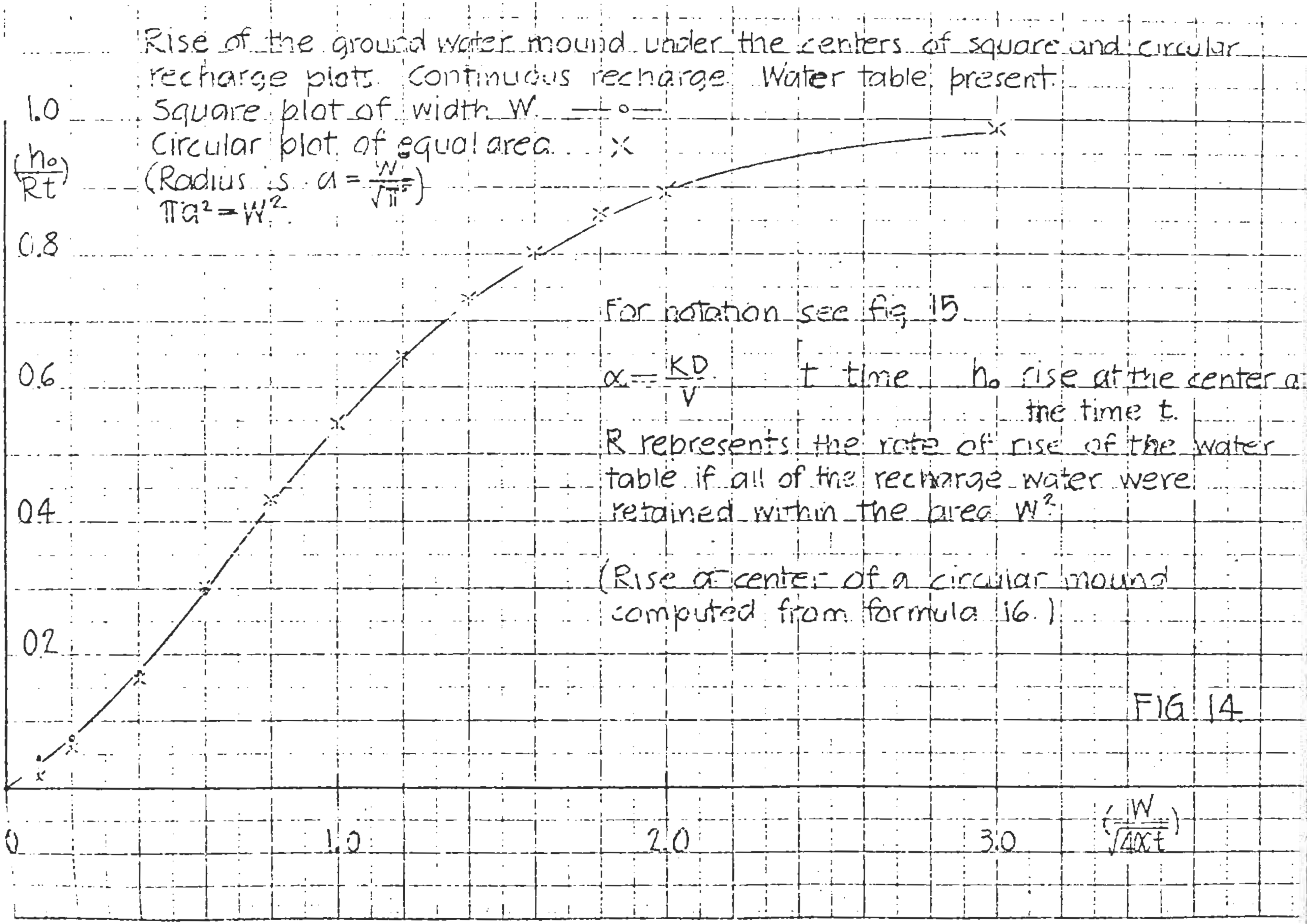
$$4\alpha t = 588900 \qquad \alpha = \frac{588900}{(4)(1296000)} = 0.114\ \mathrm{ft^2/sec}$$

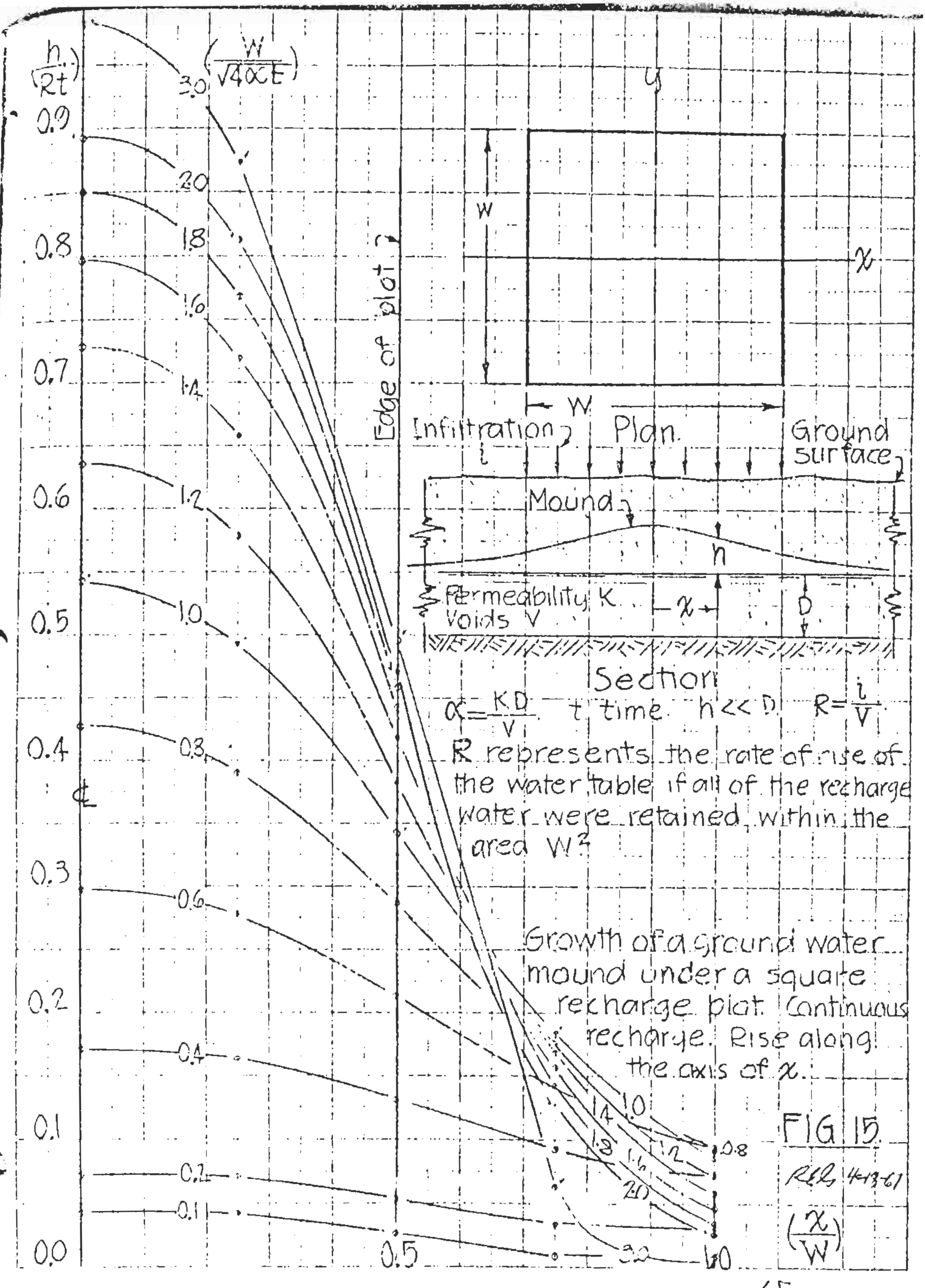
This figure is to be compared to $\alpha = 0.1\ \mathrm{ft^2/sec.}$ with which we began since the second computation retraced the ground covered in the first computation. The difference in the values is due to errors introduced in reading the charts.

The chart of figure 16 is similar to that of figure 15 but applies to a rectangular recharge plot whose length is twice its width. As before, the chart applies to the conditions along the axis of x. The pattern is symmetrical with respect to the y axis. The chart applies to the half of the pattern to the right of the y axis.

The chart of figure 17 applies at the center of a rectangular recharge plot. The pattern of rise is shown for (L/W) ratios of 1, 2, 4 and for an infinitely long strip. As the (L/W) ratio grows large compared to unity the pattern of the ground water mound, taken along a section joining the centers of the long sides, approaches the pattern which develops along a transverse section of an infinitely long strip. The correspondence is close even for $(L/W) = 4$. With $W = 330$ feet, an (L/W) ratio of 4 and the aquifer properties of the example of page 62, the pattern of the mound at the end of one week of recharge would be almost indistinguishable from that of an infinitely long strip. After 15 days there would be some differences. If, therefore, observations are made along a line joining the centers of the long sides of a rectangular recharge plot the charts for (L/W) ratios of 1, 2 and ∞ will serve fairly well for determination of aquifer properties and the delineation of mound patterns.

As an example of the use of figure 16 the shape of the ground water mound after recharge has been applied for several periods of time will be computed. The conditions will be the same as for the example of page 62 except that $(L/W) = 2$. The recharge plot would then be 660 feet long and 330 feet wide. The manner of making the computations is shown in the following tabulation for time 1 day or 86400 seconds.


Rise of the ground water mound under the centers of square and circular recharge plots. Continuous recharge. Water table present.
Square plot of width W.
Circular plot of equal area.
(Radius is $a = \frac{W}{\sqrt{\pi}}$)
$\pi a^2 = W^2$
1.0
$\left(\frac{h_o}{Rt}\right)$
0.8
0.6
0.4
0.2
0
1.0
2.0
3.0
$\left(\frac{W}{\sqrt{4\alpha t}}\right)$
For notation see fig. 15
$\alpha = \frac{KD}{V}$
t time
h_o rise at the center at the time t.
R represents the rate of rise of the water table if all of the recharge water were retained within the area W^2.
(Rise at center of a circular mound computed from formula 16.)
FIG 14



$\left(\frac{h}{Rt}\right)$
$\left(\frac{W}{\sqrt{4\alpha t}}\right)$
Edge of plot
Plan.
Infiltration i
Ground surface
Mound
Permeability K
Voids V
Section
$\alpha = \frac{KD}{V}$ t time $h \ll D$ $R = \frac{i}{V}$
R represents the rate of rise of the water table if all of the recharge water were retained within the area W^2
Growth of a ground water mound under a square recharge plot. Continuous recharge. Rise along the axis of x.
$\left(\frac{x}{W}\right)$
REG 4-13-61

FIG 15.

The (h/Rt) values are read from the chart of figure 16 for $(W/\sqrt{4\alpha t}) = 1.775$.

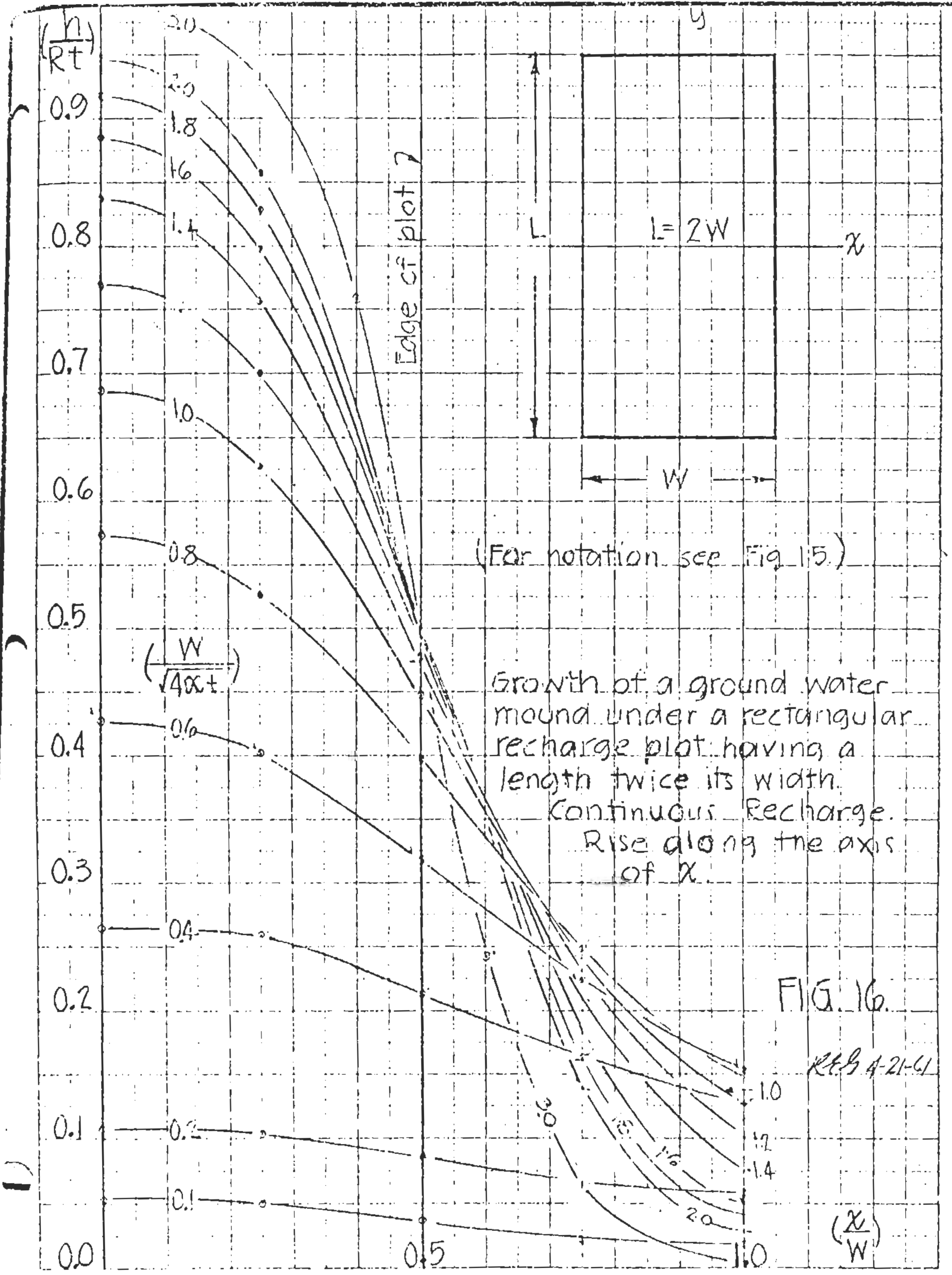
$R = 77.16 \times 10^{-6}$ ft/sec $\qquad Rt = (77.16)(10)^{-6}(86400) = 6.667$ ft.

$\sqrt{4\alpha t} = \sqrt{(4)(0.1)(86400)} = 185.9$ ft. $\qquad (W/\sqrt{4\alpha t}) = 1.775$

$\frac{x}{a}$	$\frac{h}{Rt}$	$h = \left(\frac{h}{Rt}\right) Rt$ (feet)
0.00	0.913	6.09
0.25	0.823	5.49
0.50	0.500	3.33
0.75	0.123	0.82
1.00	0.045	0.30

Robert E. Glover

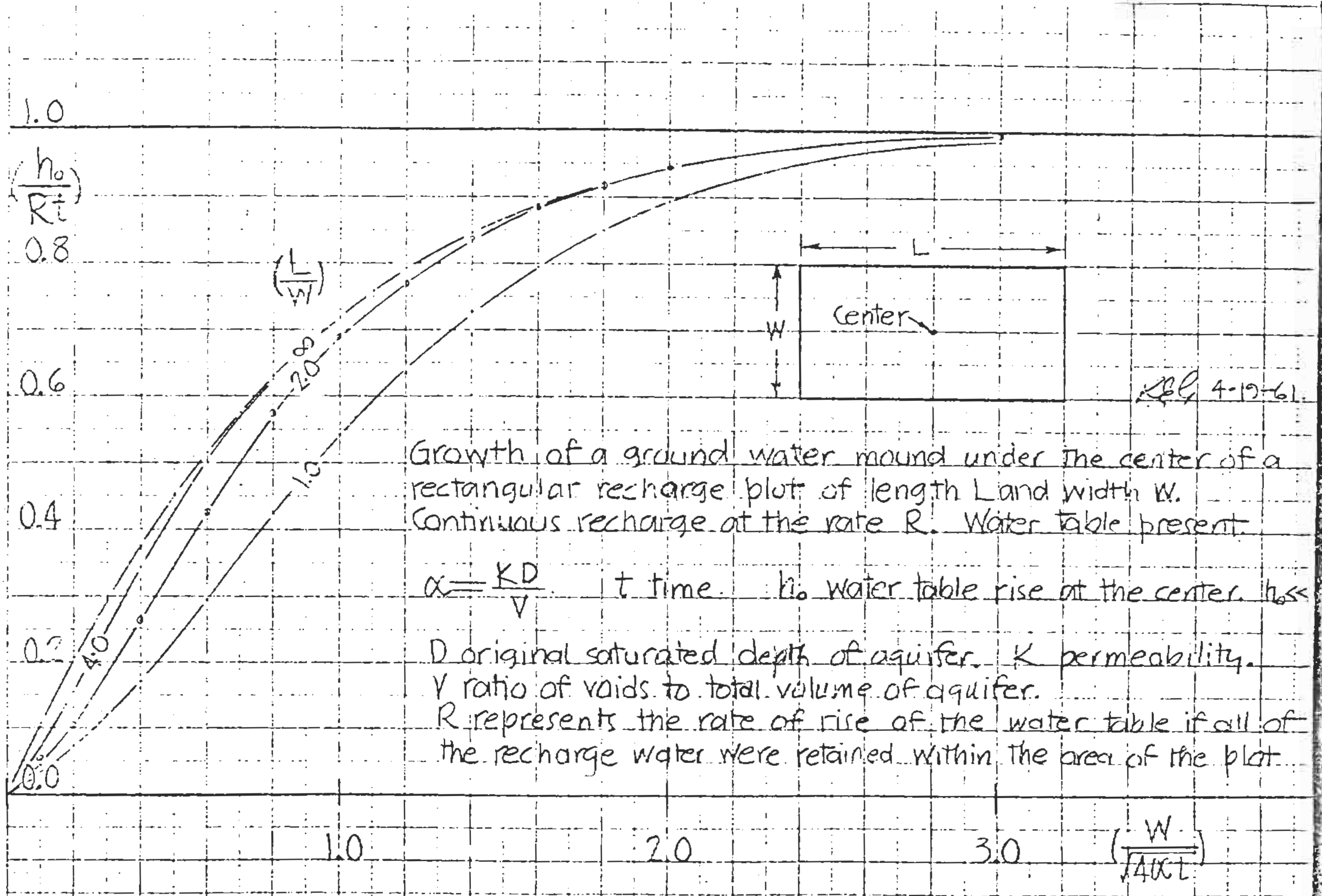
Robert E. Glover
April 25, 1961



Growth of a ground water mound under a rectangular recharge plot having a length twice its width.
Continuous Recharge.
Rise along the axis of x.

FIG. 16.

REB 4-21-61



1.0
$\left(\frac{h_o}{Rt}\right)$
0.8
0.6
0.4
0.2
0.0
$\left(\frac{L}{W}\right)$
∞
2.0
1.0
4.0
L
W
Center
KEL 4-19-61
Growth of a ground water mound under the center of a rectangular recharge plot of length L and width W. Continuous recharge at the rate R. Water table present.
$\alpha = \frac{KD}{V}$ t time. h_o water table rise at the center. $h_o \ll$
D original saturated depth of aquifer. K permeability.
V ratio of voids to total volume of aquifer.
R represents the rate of rise of the water table if all of the recharge water were retained within the area of the plot.
1.0
2.0
3.0
$\left(\frac{W}{\sqrt{4\alpha t}}\right)$

FIG 17

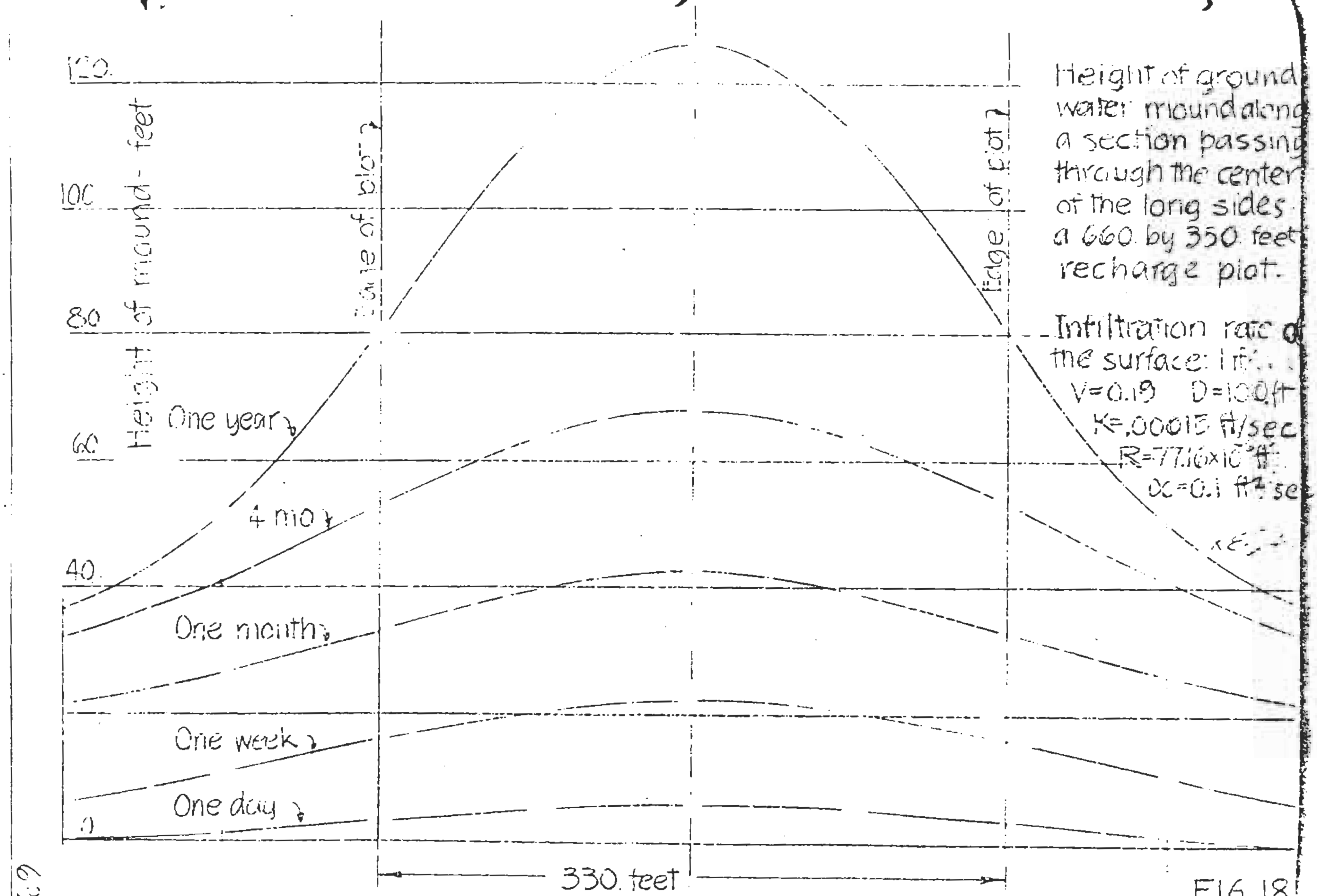

Height of ground water mound along a section passing through the center of the long sides a 660 by 350 feet recharge plot.
Infiltration rate of the surface: 1 ft
V=0.19 D=100 ft
K=.00015 ft/sec
R=77.16x10 ft
α=0.1 ft²/sec
120
100
80
60
40
0
Height of mound - feet
Edge of plot
Edge of plot
One year
4 mo
One month
One week
One day
330 feet

FIG. 18

Line. Water table initially absent. Impermeable perching layer. Transient case.

The case treated here may be described as follows: A continuous recharge along a line, of amount q_1 ft^2/sec, moves downward until it meets the surface of a horizontal impermeable perching layer. The accumulating recharge builds a mound of height h on the perching layer. The height h is a function of x, the distance from the toe of the mound, and t the time since recharge began. The bed above the perching layer has a permeability K and a voids ratio V.

Since this case is definitely of the non-linear type its solution will be sought by an iteration process to be described later. The condition of continuity is expressed by the differential equation:

$$K \frac{\partial}{\partial x}\left(h \frac{\partial h}{\partial x}\right) = V \frac{\partial h}{\partial t} \quad \ldots (92)$$

A characteristic length is:

$$\beta_1 = \frac{q_1}{K} \quad \ldots (93)$$

Let

$$\eta_1 = \frac{h}{\beta_1} \qquad \xi_1 = \frac{x}{\beta_1} \qquad \theta_1 = \frac{Kt}{V\beta_1} \quad \ldots (94)$$

In terms of these variables the differential equation becomes

$$\eta_1 \frac{\partial^2 \eta_1}{\partial \xi_1^2} + \left(\frac{\partial \eta_1}{\partial \xi_1}\right)^2 = \frac{\partial \eta_1}{\partial \theta_1} \quad \ldots (95)$$

This is free of dimensioned quantities and indicates that one chart of the form;

$\eta_1(\xi_1, \theta_1)$ will suffice for all cases.

The iteration procedure is based upon the following considerations. The flow F across a section at x at the time t is

$$F = K h \frac{\partial h}{\partial x} \quad \ldots (96)$$

In terms of the variables (94) this relation takes the form:

$$F = q_1 \eta_1 \frac{\partial \eta_1}{\partial \xi_1} \quad \ldots (97)$$

and by integration

$$\frac{\eta_1'}{q} = \int \frac{F}{q_1}\, d\xi_1 \qquad \ldots (98)$$

The flow F is used in building the mound. This relationship can be expressed in the form:

$$F = V \int \frac{\partial h}{\partial t}\, dx \qquad \ldots (99)$$

Or, in terms of the variables (94)

$$F = q_1 \int \frac{\partial \eta_1}{\partial \theta_1}\, d\xi_1 \qquad \ldots (100)$$

The iteration procedure can now progress by the following steps:

(1) Construct, by some means, a first approximation chart. (It will be shown later how this was done by considering the mound triangular).

(2) Plot from this chart a chart with η_1 as ordinate and θ_1 as abscissa. This chart will have a family of curves corresponding to equally spaced values of ξ_1.

(3) From this chart pick values of $\partial\eta_1/\partial\theta_1$ and construct from them a chart with $\partial\eta_1/\partial\theta_1$ as ordinate and ξ_1 as abscissa. The curves of this chart will correspond to chosen values of θ_1.

(4) From the values of this chart compute

$$\frac{F}{q_1} = \int \frac{\partial \eta_1}{\partial \theta_1}\, d\xi_1$$

by use of formula (100).

(5) Substitute these values in formula (98) and compute η_1.

(6) Use these values to construct a second approximation $\eta_1(S_1, \theta_1)$ chart and repeat the above process to obtain a third approximation.

(7) Repeat as many times as necessary until the nth and n + 1st approximations agree. This is the solution sought.

The accuracy obtainable by the graphical procedure outlined above is limited by the inaccuracies introduced in step 3 since the process of judging gradients by eye is not highly accurate. However, in view of the irregularities of real aquifers and the difficulty of evaluating these properties, a chart can be constructed in this way which should be accurate enough to serve well in actual practice. The iteration process could probably be carried through by a numerical procedure to obtain greater accuracy, if this were found to be desirable, but the computations required would be both exacting and tedious.

The iteration procedure can be much shortened if a good first approximation is available. Such an approximation was obtained in the present case by assuming the mound to be triangular with a base width of 2b and a height at the apex of h_a. By using the relations that the gradient (h_a/b) must be able to transmit the flow $(q_1/2)$ to each side or

$$\frac{K h_a^2}{b} = \frac{q_1}{2} \qquad \ldots (101)$$

and that the volume V_m of the mound

$$V_m = \frac{b\, h_a\, V}{2} \qquad \ldots (102)$$

must be consistant with the volume of the recharge water or

$$V_m = \frac{q_1\, t}{2} \qquad \ldots (103)$$

Elimination of b from these equations will yield

$$h_a = \sqrt[3]{\frac{q_1^2\, t}{2\, KV}} \qquad \ldots (104)$$

and, by substitution

$$b = \sqrt[3]{\frac{2K\, q_1\, t^2}{V^2}} \qquad \ldots (105)$$

The first approximation chart can be constructed from these two equations since the mound is assumed to be bounded by straight lines. The first approximation is indicated on figure 19 by dotted lines. The two shown are for $(\theta_1 / 10^6)$ equal to 1.0 and 2.0. Limits were not given for the integrals of equations (99) and (100) since they depend upon the position chosen for the origin. In making the second approximation computations it was found desirable to shift the origin to the point of recharge. The integrations were made numerically by Simpson's rule or average ordinate procedures. The first approximation described above proves to be near to the true solution. For this reason and because of the difficulties inherent in step 3 the computations were carried only to the second approximation.

Example

The bed of a previously dry water course is to be used for recharge purposes by releasing a flow of water down it. The wetted perimeter will be maintained at an average width of about 20 feet and the rate of infiltration will be 0.5 feet of water per day. The sands underlying the water course have a permeability $K = 0.0020$ ft/sec and a voids ratio of $V = 0.22$. These sands rest on an impermeable shale at a depth of 30 feet. Compute the height of the mound under the water course and the width of the mound after one month of recharge.

Solution

There are 86400 seconds in a day. Then

$$q_1 = \frac{(20)(0.5)}{86400} = .000,115,75 \text{ ft}^2/\text{sec.}$$

$$\beta_1 = \frac{q_1}{K} = \frac{.00011575}{.0020} = .05787 \text{ ft.}$$

One month is 2628000 seconds. Then

$$\theta_1 = \frac{K\,t}{V\,\beta_1} = \frac{(.0020)(2628000)}{(0.22)(.05787)} = \frac{5256}{.01273} = 412900.$$

From the chart of figure 19, by interpolation

$$\eta_1 = 60.1 \quad \text{and} \quad \xi_1 = 6650$$

Then, by using the relations of (74)

$$h = \beta_1 \eta_1 = (.05737)(60.1 = 3.48 \text{ feet}$$

$$x = \beta_1 \xi_1 = (.05737)(6650) = 385 \text{ feet}$$

The total width of the mound is then

$$(2)(385) = 770 \text{ feet.}$$

As a check, we may compare the amount of recharge supplied in the month with the contents of the mound considered to be triangular in shape.

The total supply is:

(.000,115,75)(2623000) = 304.2 cubic feet per foot of length of the watercourse.

The contents of the mound are, approximately:

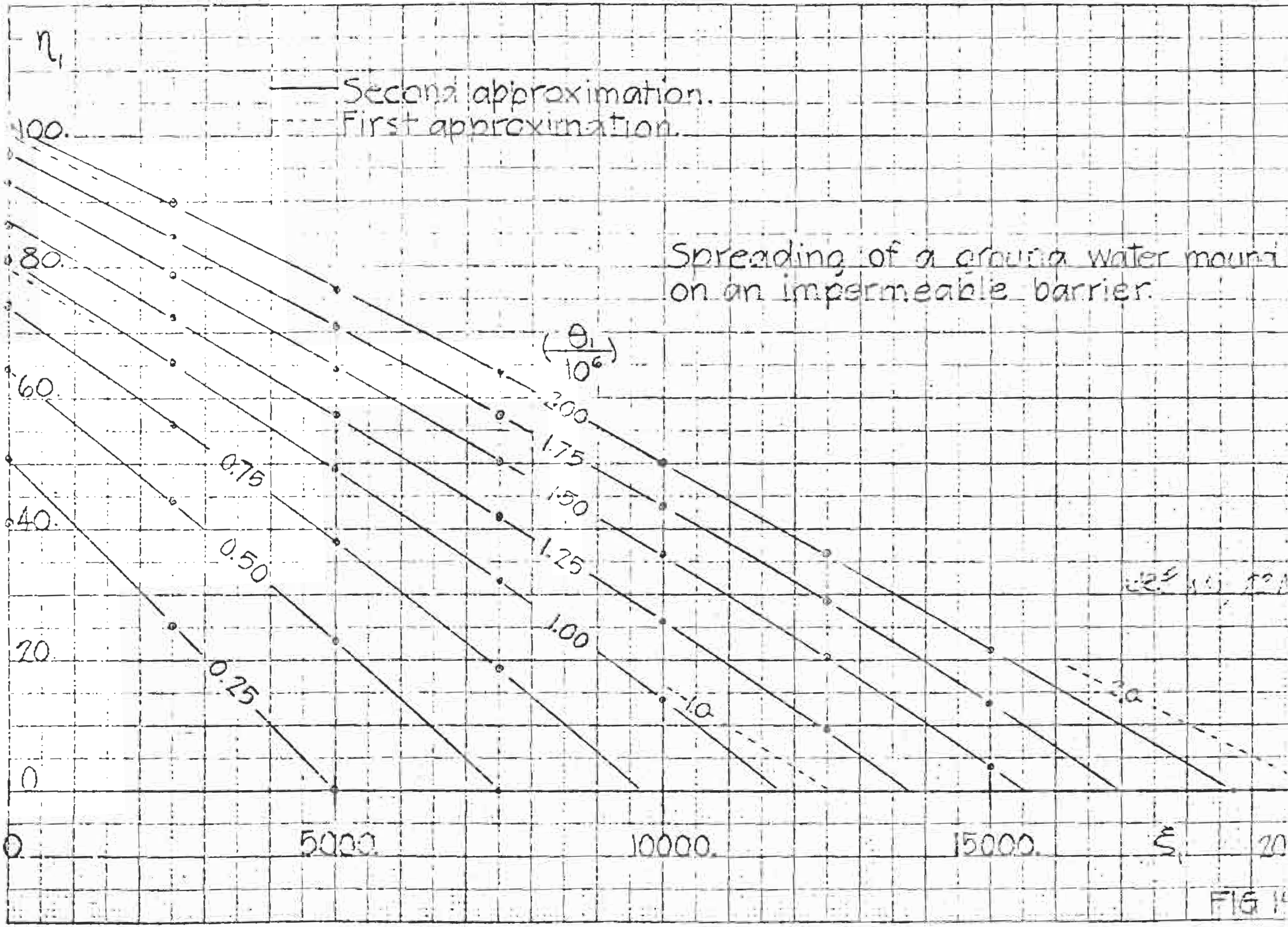
$$\frac{(770)(3.48)(0.22)}{2} = 295 \text{ cubic feet per foot of length}$$

of water course. This figure is a little low since the mound is slightly convex. The discrepancy is about 3 percent.

The correspondence is considered satisfactory.

Robert E. Glover

Robert E. Glover
May 23, 1961


η_1
Second approximation.
First approximation.
Spreading of a ground water mound on an impermeable barrier.
$\left(\frac{\theta_1}{10^6}\right)$
2.00
1.75
1.50
1.25
1.00
0.75
0.50
0.25
1.0
2.0
100.
80.
60.
40.
20.
0
0
5000.
10000.
15000.
ξ_1
20
FIG 1

References

(1) Studies of Ground Water Movement. Technical Memorandum No.657. U.S. Bureau of Reclamation, Denver, Colorado. March 1960.

(2) Source Material for a Course in Transient Ground Water Hydraulics. Compiled by R.E. Glover and M.W. Bittinger- Civil Engineering Section, Colorado State University, Fort-Collins, Colorado. April 1959.

(3) Electric Analogies and Electronic Computers- A Symposium. Paper 2569 Transactions ASCE. 1953.

(4) River Depletion Resulting from Pumping a Well Near a River. R.E. Glover and Glenn G. Balmer. Transactions AGU., Vol 35 No.3. June 1954.

(5) The Conduction of Heat. Carslaw.- MacMillan Co. 1921.

(6) Heat Conduction. L.R. Ingersoll, O.J. Zobel and A.C. Ingersoll.- McGraw-Hill. 1948

(7) Conduction of Heat in Solids. H.S. Carslaw and J.C. Jaeger.- Oxford. 1948.

(8) The Mathematics of Diffusion. J. Crank.- Oxford. 1957. (Treats many non-linear partial differential equations)

(9) Differential Equations. R.P. Agnew.- McGraw-Hill. 1942. (Chapter 16 describes Picard's method as applied to ordinary differential equations.)

(10) Memoire sur la Theorie des Equations aux Derivees Partielles et la Methode des Approximations Successives. Emile Picard.- Liouville's Journal de Mathematiques, Series 4 Vol .6. 1890- pp 145-210.

Tables

Tables of the Error Function and its Derivative. U.S. Department of Commerce- National Bureau of Standards, Applied Mathematics Series 41. 1954.
(Contains a 15 place table of the Probability Integral)

Tables of Functions and Zeros of Functions. National Bureau of Standards, Applied Mathematics Series 37. 1954.
(Contains a table of the exponential integral $E_1(x)$.)

Tables of Sine, Cosine and Exponential Integrals. National Bureau of Standards, Mathematical Tables MT 5 and MT 6. 1940.

A Short Table of Integrals. B.O. Peirce.-Ginn & Co. 1929.
(Contains tables of the Probability Integral and a table of Natural Logarithms)

Heat Conduction. (Reference 6)
(Contains a table of the Probability Integral and a table of the integral

$$\int_x^\infty \frac{e^{-\beta^2}}{\beta}\,d\beta$$

Source Material for a Course in Transient Ground Water Hydraulic (Reference 2)
(Contains a table of the line source integral

$$\sqrt{\pi}\int_x^\infty \frac{e^{-u^2}}{u^2}\,du$$

Smithsonian Mathematical Tables. Hyperbolic Functions.
(Contains a 7 place table of the exponential function e^x)

Mathematical Tables. H.B. Dwight.- Dover. 1958
(Contains tables of the Exponential function, Exponential integ and Probability integral.)

Some Applications in Physics of the P Function. Joseph I. Master Journal of Chemical Physics- Vol 23 No. 10. October 1955.
(Contains tables of the P function).

Indox

Page

Index [illegible] Page

 Page