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A PERTURBATION SOLUTION OF THE NONLINEAR BOUSSINESQ EQUATION: THE CASE OF CONSTANT INJECTION INTO A RADIAL AQUIFER

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ABSTRACT

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A generalized version of the Boussinesq equation is solved, using a singular perturbation technique. Flow takes place in radial directions into an initially-dry homogeneous isotropic soil under the assumption of a constant injection rate at the origin. Explicit formulae are derived for the location of the advancing wetting front and the free surface height (or pressure head). Analytical results are compared graphically with a numerical finite-element solution. Excellent agreement is obtained between the results of the two methods.

INTRODUCTION

The Boussinesq equation is widely used in studying groundwater flow problems. It is based upon the Dupuit—Forchheimer (D—F) assumption that the groundwater, when bounded above by a gently sloping phreatic (free) surface, moves essentially horizontally (Bear, 1972, Ch. 8). This equation has remained a powerful tool in modelling and analyzing flow patterns associated with many aspects of subsurface flows in unconfined aquifers. Recent concern about environmental quality in relation to the transport of dissolved substances by groundwater has increased the importance attached to this equation.

The basic equation is nonlinear. The nonlinearities, however, are not strong, appearing only in the quadratic degree. Until now, solutions for this equation have relied heavily on numerical schemes and linearization devices. Numerical solutions by Moody (1966), Zucker et al. (1973), Dass and Morel-Seytoux (1974), Skaggs (1975), and Gureghian (1978); or works on linearization techniques by Maasland and Bittinger (1963) and Bear et al. (1968), are examples of this kind of approach. On the other hand, very few analytical solutions are available (see, e.g., Van Schilfgaarde, 1963, 1964).

The phenomena of moisture absorption by unsaturated soils, heat conduction in nonlinear media, and the general groundwater movement under the D—F assumptions are all described by the same nonlinear diffusion equation. Babu (1976a, b, c) initiated and developed a general method of solution to the problems described by such a nonlinear diffusion equation. Techniques of perturbation analysis formed the basis of this method.

A nonlinear diffusion problem (in one-dimensional form) wherein diffusivity varied as a positive power of water saturation, was solved by Babu and Van Genuchten (1979a) by means of singular perturbation techniques. A similar procedure is adopted in the following sections to solve the nonlinear diffusion problem in three-dimensional space. The flows are assumed to be symmetrical about the vertical axis, and again, the diffusivity is taken to vary as a positive power of the concentration (temperature, water saturation, etc.). Solutions to the Boussinesq equations follow as special cases of the general solution.

The work presented here is the result of attempts to derive analytical forms of solutions to the Boussinesq equation in three-dimensional space. Analytical expressions and formulae become very useful in checking the accuracy of numerical schemes of solution. Furthermore, the behaviour of solutions at large times and at large distances, is described more easily by analytical solutions than by numerical solutions. An outstanding feature of the method presented below is that the singular terms, that so often lead to a breakdown of the formulae connected with such perturbation schemes, are eliminated from the solution. Thus, the method of solution yields results that remain valid for all times and distances. Agreement between the perturbation solution, as presented here, and a finite-element numerical solution, was excellent.

In what follows, a constant-flux boundary condition is assumed to be in existence at the origin. Other types of boundary conditions are possible for the general Boussinesq equation; however, in this analysis, no other boundary conditions are considered. The perturbation method is well suited for handling constant saturation, and constant flux, boundary conditions, along with zero-state initial conditions. Non-zero initial conditions, and time-dependent boundary conditions, would require more elaborate and involved analysis (Babu and Van Genuchten, 1979b). Therefore, no attempt was made to include such cases here.

THE EQUATIONS

All variables in this study are assumed to have been rendered dimensionless by appropriate normalizations. Radial distance is denoted by r, time by t, and the pressure head by h. For the special case of interest here, one may interpret h as the free surface elevation in an unconfined aquifer. For unsaturated soils absorbing mositure, h is identifiable with the moisture content. In the heat conduction problems, h may be taken as the temperature.

If it is assumed that the flux law is given by:

$$v = -h^n(\delta h/\delta r) \qquad (n > 0)$$

the equation of continuity takes the form of:

Partial differential equation:

$$\frac{1}{r}\frac{\delta}{\delta r}\left(rh^n\frac{\delta h}{\delta r}\right) = \delta h/\delta t \qquad n > 0, \qquad 0 < r < r_f(t) \tag{a}$$

The location of the wetting front is given by $r_f(t)$; it is an unknown function of time, and marks the extent to which water has penetrated at any given time t. The determination of $r_f(t)$ constitutes a part of the solution process.

The initial condition is one of zero state:

$$r_f(0) = 0; \qquad h = 0, \qquad t = 0 \tag{b}$$

The boundary conditions at the *front* are taken as zero water pressure, and zero water flux:

$$h = 0. \qquad r = r_f(t) \qquad \text{(unknown)} \tag{c}$$

and

$$h^n\frac{\delta h}{\delta r} = 0, \qquad r = r_f(t) \qquad \text{(unknown)} \tag{d}$$

The boundary condition at the *origin* is one of constant flux (constant injection rate):

$$\lim_{r \to 0} 2\pi r h^n \frac{\delta h}{\delta r} = -Q(n) \qquad \text{(given)} \tag{e}$$

Although it is assumed that the flux Q is specified externally, it should be obvious that no water can be transported if n becomes large. Therefore, on the right-hand-side of eq. e, the imposed flux $Q(n)$ is shown to depend on the index n.

Eq. a may be considered as a generalized version of the well-known Boussinesq equation. The next step is to solve the boundary value problem formulated in eqs. a–e.

THE TRANSFORMED EQUATIONS

A sequence of transformations will be introduced now. These transformations reduce the governing partial differential equation to an ordinary differential equation, and simplify to some extent the associated initial/boundary conditions. Furthermore, the original domain of flow $0 \leqslant r \leqslant r_f(t)$ will be normalized to lie between 0 and 1.

The perturbation parameter:

$$\epsilon \equiv 1/(n+1) \tag{1}$$

Reduced flux:

$$q \equiv Q(\epsilon)/4\pi\epsilon \tag{2}$$

A new dependent variable:

$$P \equiv h^{n+1}/q, \qquad \text{or,} \qquad h \equiv (Pq)^{\epsilon} \tag{3}$$

A new independent variable x, with range $0 \leqslant x \leqslant 1$:

$$x = (r^2/4t) \cdot [\epsilon\Phi(\epsilon)q^{1-\epsilon}]^{-1} \tag{4}$$

where $\Phi(\epsilon)$ is an unknown function (of the parameter ϵ) such that $x = 1$ defines the wetting front position:

$$r_f^2(t) = 4t \cdot \epsilon \cdot \Phi(\epsilon) \cdot q^{1-\epsilon} \tag{5}$$

with

$$\Phi(\epsilon) \equiv 1 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots \qquad (\phi_1, \phi_2, \ldots, \text{are unknown constants}) \tag{5a}$$

It may be mentioned in passing that P, as defined in identity (3), is usually associated with the term Kirchhoff Potential of Soil Physics and Heat Conduction literature.

In terms of these new variables, eqs. a—e transform into:

The differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}P}{\mathrm{d}x}\right) + \Phi(\epsilon) \cdot x\frac{\mathrm{d}P^{\epsilon}}{\mathrm{d}x} = 0 \qquad 0 < x < 1 \tag{6}$$

The boundary conditions at the front:

$$P = 0, \qquad x = 1 \tag{7}$$

$$\mathrm{d}P/\mathrm{d}x = 0, \qquad x = 1 \tag{8}$$

The flux condition at the origin:

$$\lim_{x \to 0} x\frac{\mathrm{d}P}{\mathrm{d}x} = -1 \tag{9}$$

Note: From a mathematical point of view, conditions (7)—(9) constitute an overspecification for the second-order ordinary differential equation. However, eq. 6 contains the function $\Phi(\epsilon)$ associated with the unknown location of the wetting front. Since an additional unknown is introduced into the problem via the function $\Phi(\epsilon)$, a third condition becomes necessary to complete the formulation of the problem.

It will be seen later (see comment after eq. 32) that condition (8) is difficult to use in the solution scheme, because of certain indeterminate and unbounded quantities appearing in eq. 32. To circumvent this difficulty, we integrate eq. 6 by parts, and use conditions (7)—(9) to develop an alternative formula.

From eq. 6 and condition (8):

$$x\frac{dP}{dx} + \Phi(\epsilon)\cdot\int_1^x x\frac{dP^\epsilon}{dx}\,dx = 0$$

Use of condition (9) gives:

$$-1 + \Phi(\epsilon)\cdot\int_1^0 x\frac{dP^\epsilon}{dx}\,dx = 0$$

Integration by parts yields:

$$1 = \Phi(\epsilon)\left[x\cdot P^\epsilon\Big|_{x=1}^{x=0} - \int_1^0 1\cdot P^\epsilon\cdot dx\right]$$

Use of condition (7) in the above relation finally gives:

$$\Phi(\epsilon)\cdot\int_0^1 [P(x)]^\epsilon\,dx = 1 \tag{10}$$

Alternatively, eq. 10 may be derived as follows. The condition of constant flux, (e), and the increase in the total mass of water in the aquifer, lead to the following equation (θ is the water content):

$$\int_0^{r_f(t)} \theta(h)\cdot 2\pi r dr = \int_0^t Q\cdot dt = Q\cdot t \tag{11}$$

Now

$$Q = 4\pi q\epsilon \tag{2}$$

$$\theta(h) \equiv h = (Pq)^\epsilon \tag{3a}$$

and from eq. 4:

$$2\pi r dr = 4\pi t\cdot\epsilon\cdot\Phi(\epsilon)\cdot q^{1-\epsilon}\cdot dx \tag{4a}$$

Use of eqs. 2, 3a, 5 and 4 in eq. 11 yields ($r = r_f(t), x = 1$):

$$\int_{x=0}^{x=1} (Pq)^\epsilon 4\pi t\cdot\epsilon q^{1-\epsilon}\cdot\Phi(\epsilon)\cdot dx = (4\pi q\epsilon)t$$

Minor cancellations result again in:

$$\Phi(\epsilon)\cdot\int_0^1 P^\epsilon\cdot dx = 1 \tag{10}$$

Therefore, the alternate condition (10) merely states that the total mass of accumulated water in the aquifer equals the total mass of water injected at the origin during the time t.

A solution to eq. 6 with conditions (7), (9) and (10) will be given in the next section.

THE PERTURBATION SCHEME

The solution is assumed to take the form:

$$P \equiv P(x;\epsilon) = \exp\left[(\epsilon + \epsilon^2\mu_1 + \epsilon^3\mu_2 + \ldots)\left(\int_0^x \mathrm{d}x/\ln x\right)\right] \cdot$$

$$[(-\ln x) + \epsilon P_1(x) + \epsilon^2 P_2(x) + \ldots] \tag{12}$$

where $\mu_1, \mu_2, \ldots$, are unknown constants, and $P_1(x), P_2(x), \ldots$, are unknown functions, to be determined next. The above structure for the solution, especially the exponential factor, was suggested by rewriting eq. 6 in the form:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x\frac{\mathrm{d}P}{\mathrm{d}x}\right) + x\frac{\mathrm{d}P}{\mathrm{d}x}\epsilon\Phi(\epsilon)P^{\epsilon-1} = 0$$

dividing out by $(x \cdot \mathrm{d}P/\mathrm{d}x)$, and finally integrating once with respect to x in an asymptotic sense near $x = 0$. This term would also be the starter term for many rapidly convergent iterative schemes aimed at solving such nonlinear problems.

Since eq. 12 is taken as the solution, it must satisfy the differential equation (6), and conditions (7), (9) and (10). At the same time, it is expected that functions $P_1(x), P_2(x), \ldots$, are bounded in the entire interval $0 \leqslant x \leqslant 1$.

As a next step in the solution, the power series expansions of eq. 12 in parameter ϵ are written down:

$$P(x;\epsilon) = (-\ln x) + \epsilon(P_1 - i\ln x) + \epsilon^2\left(P_2 + iP_1 - \frac{i^2}{2}\ln x - \mu_1 i\ln x\right) + \ldots \tag{13}$$

and

$$[P(x;\epsilon)]^{\epsilon} = 1 + \epsilon\ln(-\ln x) + \epsilon^2\left[\frac{P_1}{-\ln x} + i + \frac{[\ln(-\ln x)]^2}{2}\right] + \ldots \tag{13a}$$

where

$$i \equiv i(x) = \int_0^x \frac{\mathrm{d}y}{\ln y}, \qquad P_1 \equiv P_1(x) \qquad \text{and} \qquad P_2 \equiv P_2(x) \tag{14}$$

Although the above expansions in ϵ apparently break down near $x = 0$ and $x = 1$, it is still possible to free the original differential equation of the singularities.

THE SOLUTION

Eqs. 13 and 13a are substituted into eq. 16 and conditions (7), (9) and (10). Expansion (5a) is also utilized at this stage. Collection of like powers of ϵ from these equations leads to the following sequence of linear problems.

ε-terms:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\left(\frac{\mathrm{d}P_1}{\mathrm{d}x}-\frac{1}{x}\int_0^x\frac{\mathrm{d}y}{\ln y}-1\right)\right]+x\frac{\mathrm{d}}{\mathrm{d}x}[\ln(-\ln x)] = 0 \tag{15}$$

$$P_1 = 0, \qquad x = 1 \tag{16}$$

$$x\,\mathrm{d}P_1/\mathrm{d}x = 0, \qquad x = 0 \tag{17}$$

and

$$\int_0^1 [\phi_1 + \ln(-\ln x)]\,\mathrm{d}x = 0 \tag{18}$$

ϵ^2-terms:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{\mathrm{d}P_2}{\mathrm{d}x}+x\frac{\mathrm{d}P_1}{\mathrm{d}x}\left(\int_0^x\frac{\mathrm{d}y}{\ln y}\right)+\frac{xP_1}{\ln x}-\frac{1}{2}\left(\int_0^x\frac{\mathrm{d}y}{\ln y}\right)^2-x\int_0^x\frac{\mathrm{d}y}{\ln y}\right.$$
$$\left.-\mu_1\int_0^x\frac{\mathrm{d}y}{\ln y}-\mu_1 x\right]+x\left[\phi_1\frac{\mathrm{d}}{\mathrm{d}x}[\ln(-\ln x)]\right.$$
$$\left.+\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{-P_1}{\ln x}+\frac{[\ln(-\ln x)]^2}{2}+\int_0^x\frac{\mathrm{d}y}{\ln y}\right)\right] = 0 \tag{19}$$

$$P_2 = 0, \qquad x = 1 \tag{20}$$

$$x\cdot\mathrm{d}P_2/\mathrm{d}x = 0, \qquad x = 0 \tag{21}$$

and

$$\int_0^1 [\phi_2+\phi_1 F_1(x)+F_2(x)]\,\mathrm{d}x = 0 \tag{22}$$

with

$$F_1(x) \equiv \ln(-\ln x) \tag{23}$$

$$F_2(x) \equiv (1-x)/\ln x + \tfrac{1}{2}[\ln(-\ln x)]^2 + \int_0^x \mathrm{d}y/\ln y \tag{24}$$

And so on, for higher-order terms in ϵ^2, ϵ^3, ...

Evaluation of $\mathrm{P}_1(\mathrm{x})$ *and the constant* ϕ_1

Solving eqs. 15—18 results in:

$$P_1(x) = x-1 \tag{25}$$

$$\phi_1 = -\int_0^1 \ln(-\ln x)\,\mathrm{d}x \simeq 0.5775 \text{ (Euler's constant)} \tag{26}$$

Evaluation of $P_2(x)$ *and constants* μ_1 *and* ϕ_2

Rewriting eq. 19 and simplifying to some extent with use of eqs. 25 and 26 leads to:

$$\frac{d}{dx}\left(x\frac{dP_2}{dx}\right)+\frac{2(x-1)+1+\phi_1-\mu_1+\ln(-\ln x)-\int_0^x dy/\ln y}{\ln x}=\mu_1 \qquad (27)$$

Or, equivalently, integrating by parts the expression $\int_0^x dy/\ln y$:

$$\frac{d}{dx}\left(x\frac{dP_2}{dx}\right)+\frac{2(x-1)+(1-x)\ln(-\ln x)+1+\phi_1-\mu_1+\int_0^x \ln(-\ln y)\,dy}{\ln x}$$

$$=\mu_1 \qquad (28)$$

If singularities at $x = 1$ are to be avoided in eq. 28, it is necessary that:

$$1+\phi_1-\mu_1+\int_0^1 \ln(-\ln x)\,dx = 0 \qquad (29)$$

This and eq. 18 determine the constant μ_1:

$$\mu_1 = 1+\phi_1+\int_0^1 \ln(-\ln x)\,dx = 1 \qquad (30)$$

Thus, eq. 28 reduces to:

$$\frac{d}{dx}\left(x\frac{dP_2}{dx}\right)+\frac{[2(x-1)+(1-x)\ln(-\ln x)-\int_x^1 \ln(-\ln y)\,dy]}{\ln x}=1 \qquad (31)$$

Integration of eq. 31, utilizing conditions (20) and (21) finally gives:

$$P_2(x) = (x-1)+(x-1)^2+(\ln x)\int_0^x \frac{2(1-x)+\int_x^1 (1-y)/y\ln y\cdot dy}{\ln x}\,dx$$

$$+\int_x^1 \frac{1-y}{\ln y}\,dy - x\int_x^1 \frac{1-y}{y\ln y}\,dy \qquad (32)$$

Expressions suitable for computational detail have been selected in structuring $P_2(x)$. Other forms of $P_2(x)$ are also possible if analytical simplicity of structure is sought.

Comment: If term-by-term differentiation is accepted in eqs. 13 and 13a, the expressions for (dP/dx) derived by using eqs. 25 and 32 in eq. 13, would contain unbounded and indeterminate quantities when evaluated at $x = 1$. Such unbounded quantities would obviously hinder the evolution of the solution. It was, therefore, necessary to replace condition (8) by an alternate expression like condition (10), etc.

Lastly, from eqs. 22—26, the constant ϕ_2 may be determined by straightforward quadratures:

$$\phi_2 = -\left(\int_0^1 F_1(x)\cdot\phi_1\,dx+\int_0^1 F_2(x)\,dx\right)\simeq\phi_1^2+0.411\simeq 0.7445 \qquad (33)$$

THE RESULTS

To summarize, a solution to the problem formulated in eqs. 6—10 is given by the following formulae (to ϵ^2-order terms):

$$P(x;\epsilon) = \exp\left[(\epsilon + \epsilon^2 + \ldots)\left(\int_0^x \frac{dy}{\ln y}\right)\right] \cdot [(-\ln x) + \epsilon P_1(x) + \epsilon^2 P_2(x) + \ldots] \tag{34}$$

where:

$$P_1(x) \equiv (x - 1) \tag{35}$$

and

$$P_2(x) \equiv x(x-1) + (\ln x)\int_0^x \frac{2(1-y) + \int_y^1 (1-t)/t\ln t \cdot dt}{\ln y}\,dy + \int_x^1 \frac{1-y}{\ln y}\,dy - x\int_x^1 \frac{1-y}{y\ln y}\,dy \tag{36}$$

Through back substitutions via eqs. 1—5a, the pressure head h is easily written down:

$$h(r,t;\epsilon) \equiv (Pq)^\epsilon = \left[\frac{Q(\epsilon)}{4\pi\epsilon}\right]^\epsilon \cdot P^\epsilon \tag{37}$$

$$P^\epsilon \equiv \exp\left[(\epsilon^2 + \epsilon^3 + \ldots)\left(\int_0^x \frac{dy}{\ln y}\right)\right] \cdot [(-\ln x) + \epsilon P_1(x) + \epsilon^2 P_2(x)]^\epsilon \tag{38}$$

where

$$\epsilon \equiv 1/(n+1); \qquad \phi_1 \simeq 0.5775; \qquad \phi_2 \simeq 0.7455; \qquad \ldots \tag{39}$$

$$r^2 = 4xt\epsilon(1 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots) \cdot [Q(\epsilon)/4\pi\epsilon]^{1-\epsilon} \tag{40}$$

$$Q(\epsilon) \equiv \text{the given constant rate of injection} \tag{41}$$

Location of the advancing wet front $r_f(t)$ is given by setting $x = 1$ in eq. 40. Thus, for the location of the wet front:

$$r_f(t) = [4t\epsilon(1 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots)]^{1/2} \cdot [Q(\epsilon)/4\pi\epsilon]^{(1-\epsilon)/2} \tag{42}$$

Eqs. 35—42, therefore, constitute the solution to the problem formulated in eqs. a—e at the beginning of this paper.

Special case of small x *(or small* r, t, *etc.)*

A simpler expression for the solution can be obtained when x (or r and t) is small. From the Appendix (A-1—A-4), if $x \simeq 0$:

$$P(x;\epsilon) \simeq \exp(0) \cdot \left[(-\ln x) + \epsilon(x-1) - \epsilon^2[x\{2 + 2\ln x + (\ln x)^{-1}\} + 0.693]\right] \tag{43}$$

or, neglecting terms in x:

$P(x;\epsilon) \simeq -(\ln x + \epsilon + 0.693\,\epsilon^2)$

Thus, for smaller x:

$$h(r,t;\epsilon) \simeq [Q(\epsilon)/4\pi\epsilon]^{\epsilon} \cdot [-\ln x - \epsilon - 0.693\epsilon^2]^{\epsilon} \qquad (44)$$

COMPARISON EXAMPLE: THE BOUSSINESQ EQUATION

Although eqs. 34–44 are valid for all $n > 0$, if ϵ is small, accurate results are obtained with just a few terms, say $P_1(x)$ and $P_2(x)$. A case of practical importance, the Boussinesq equation in radial coordinates, is given by $n = 1$, or $\epsilon = \frac{1}{2}$. Fig. 1 compares graphically the results obtained by the methods of this paper with those generated by a numerical scheme applied to the eqs. a–e. Numerical results were obtained with a modified version of Hermitian finite-element code of Van Genuchten (1978). For this problem, n was set equal to 1 in the original eqs. a–e, while the discharge Q was given the

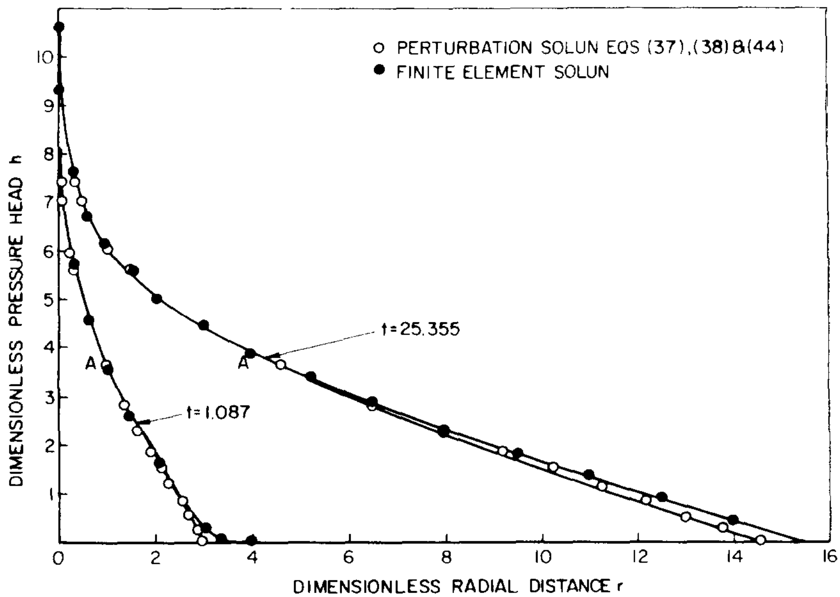

Fig. 1. Comparison between analytical perturbation solution and numerical finite-element solution for the Boussinesq equation, describing radial flows under constant injection rates. All quantities are in dimensionless form. The constant injection rate at the origin was taken as $Q = 50$ in the original eqs. a–e. The pressure head h may be directly associated with the height of the free surface in this problem. Values of h beneath the point A in the figure, where $h \leqslant 3.64$, $x \geqslant 0.1$, were computed using eqs. 37 and 38; remaining higher values of $h \geqslant 3.64$ were obtained from the asymptotic expressions in eqs. 43–44 applied to the range $0 \leqslant x \leqslant 0.1$.

value $Q(n) = 50$. Two cases were considered, one at $t = 25.355$, and the other at $t = 1.087$ so as to give a reasonable spread of the time variable. Since the perturbation solution is a similarity solution, profiles associated with all values of r and t would be generated through eqs. 24—41. For small values of x, or r, the asymptotic approximations (43) and (44) were used, thereby eliminating the need for numerical integrations. Gauss—Legendre quadrature formulae, with four Gaussian points and ten subintervals, were sufficient to evaluate all integrals and constants in eqs. 34 and 36. Fig. 1 shows that there was excellent agreement between numerical and perturbation solutions.

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APPENDIX

When $x \to 0$ in eqs. 34 and 36, the following approximations will obviate the need for numerical integrations:

(i) $$\int_0^x \frac{\mathrm{d}y}{\ln y} \simeq \left(\frac{x}{\ln x}\right) \simeq 0 \qquad \text{(A-1)}$$

(ii) $$I(x) \equiv (\ln x) \cdot \int_0^x \left(\frac{2(1-y) + \int_y^1 (1-t)/t\ln t \cdot \mathrm{d}t}{\ln y}\right) \mathrm{d}y$$
$$= (\ln x) \cdot \int_0^x G(y)\mathrm{d}y, \qquad \text{say}$$

Application of L'Hospital's rule leads to:

$$\lim_{x \to 0} I(x) = \lim_{x \to 0} \frac{\int_0^x G(y)\mathrm{d}y}{(1/\ln x)} = \lim_{x \to 0} \frac{G(x)}{-1/x(\ln x)^2}$$
$$= \lim_{x \to 0} [2x(1-x)\ln x] - \lim_{x \to 0} \left[\int_x^1 (1-t)/t\ln t \cdot \mathrm{d}t\right] \Big/ (x\ln x)$$
$$= \lim_{x \to 0} [2x(1-x)\ln x + x(1-x)]$$

Thus:

$$I(x) \simeq -x(1-x)(1+2\ln x) \simeq -x(1+2\ln x), \qquad \text{if } x \text{ is small} \qquad \text{(A-2)}$$

(iii) Similarly:

$$\lim_{x \to 0} x \cdot \int_x^1 (1-y)/y\ln y \cdot \mathrm{d}y = \lim_{x \to 0} \int_x^1 (1-t)/t\ln t \cdot \mathrm{d}t \Big/ (x^{-1})$$

leads to:

$$x\int_x^1 (1-y)/y\ln y \cdot dy \simeq x(1-x)\ln x \simeq x/\ln x, \qquad \text{when } x \text{ is small} \qquad \text{(A-3)}$$

Combining approximations (i), (ii) and (iii), and noting:

$$\int_0^1 (1-x)/\ln x \cdot dx = \ln(\tfrac{1}{2})$$

leads to a simplified version of eq. 32:

$$P_2(x) \simeq -x[2 + 2\ln x + (\ln x)^{-1}] - 0.693 \qquad \text{(A-4)}$$

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