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ECONOMICAL ALTERNATIVES TO GAUSSIAN QUADRATURE OVER ISOPARAMETRIC QUADRILATERALS

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The determination of optimal numerical integration formulas for finite element analyses is important for improving the efficiency of this method. Virtually all basic finite element texts[1-3] discuss numerical integrations in conjunction with isoparametric elements and point out the merits of using Gaussian quadrature (GQ). Indeed for integration of polynomials, GQ produces the highest order of accuracy with a given number of quadrature points of any formula. Other types of formulas have been proposed which are also used for numerical integrations.[4-6] It is the purpose of this paper to present other integration formulas which, at first sight, may appear significantly less efficient that GQ but are in fact highly competitive with GQ when applied to finite element analysis.

The non-Gaussian integration formulas for two-dimensional elements which are presented in Table I use more integration points than GQ in evaluating an integral with comparable

Table I. Numerical integration formulas with their truncation errors

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta)\,d\xi\,d\eta \simeq \sum_{i=1}^{n} w_i f(\xi_1, \eta_i) + E$$

$n=4$ (Gaussian Quadrature)		
w_i	(ξ_i, η_i)	E
1·0	$\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$	$\frac{2}{135}\left(\frac{\partial^4 f}{\partial\xi^4}+\frac{\partial^4 f}{\partial\eta^4}\right)$
$n=4$ (Trapezoidal Rule)		
w_i	(ξ_i, η_i)	E
1·0	$(\pm 1, \pm 1)$	$-\frac{4}{3}\left(\frac{\partial^2 f}{\partial\xi^2}+\frac{\partial^2 f}{\partial\eta^2}\right)$
$n=5$		
w_i	(ξ_i, η_i)	E
$\frac{5}{9}$	$\left(\pm\sqrt{\frac{3}{5}}, \pm\sqrt{\frac{3}{5}}\right)$	$-\frac{4}{45}\frac{\partial^4 f}{\partial\xi^2\,\partial\eta^2}$
$\frac{16}{9}$	$(0, 0)$	

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Table I. *(continued)*

$n=5$

w_i	(ξ_i, η_i)	E
$\frac{1}{3}$	$(\pm 1, \pm 1)$	$-\frac{1}{45}\left(\frac{\partial^4 f}{\partial \xi^4}+\frac{\partial^4 f}{\partial \eta^4}\right)-\frac{2}{9}\frac{\partial^2 f}{\partial \xi^2\,\partial \eta^2}$
$\frac{8}{3}$	$(0, 0)$	

$n=8$

w_i	(ξ_i, η_i)	E
$-\frac{1}{3}$	$(\pm 1, \pm 1)$	
$\frac{4}{3}$	$(\pm 1, 0)$	$-\frac{1}{45}\left(\frac{\partial^2 f}{\partial \xi^4}+\frac{\partial^4 f}{\partial \eta^4}\right)-\frac{4}{9}\frac{\partial^4 f}{\partial \xi^2\,\partial \eta^2}$
$\frac{4}{3}$	$(0, \pm 1)$	

$n=8$

w_i	(ξ_i, η_i)	E
$\frac{1}{6}$	$(\pm 1, \pm 1)$	$-\frac{4}{45}\frac{\partial^4 f}{\partial \xi^2\,\partial \eta^2}$
$\frac{5}{6}$	$\left(\pm\frac{1}{\sqrt{5}}, \pm\frac{1}{\sqrt{5}}\right).$	

$n=9$ (Simpson's Rule)

w_i	(ξ_i, η_i)	E
$\frac{1}{9}$	$(\pm 1, \pm 1)$	
$\frac{4}{9}$	$(\pm 1, 0)$	
$\frac{4}{9}$	$(0, \pm 1)$	$-\frac{1}{45}\left(\frac{\partial^4 f}{\partial \xi^4}+\frac{\partial^4 f}{\partial \eta^4}\right)$
$\frac{16}{9}$	$(0, 0)$	

Table I. (*continued*)

$n = 9$ (Gaussian Quadrature)

w_i	(ξ_i, η_i)	E
$\frac{25}{81}$	$\left(\pm\sqrt{\frac{3}{5}}, \pm\sqrt{\frac{3}{5}}\right)$	
$\frac{40}{81}$	$\left(0, \pm\sqrt{\frac{3}{5}}\right)$	$\frac{1}{7875}\left(\frac{\partial^6 f}{\partial\xi^6}+\frac{\partial^6 f}{\partial\eta^6}\right)$
$\frac{40}{81}$	$\left(\pm\sqrt{\frac{3}{5}}, 0\right)$	
$\frac{64}{81}$	$(0, 0)$	

$n = 9$

w_i	(ξ_i, η_i)	E
$\frac{1}{9}$	$(\pm 1, \pm 1)$	$-\frac{1}{47250}\left(\frac{\partial^6 f}{\partial\xi^6}+\frac{\partial^6 f}{\partial\eta^6}\right)$
$\frac{10}{9}$	$\left(\pm\sqrt{\frac{2}{5}}, 0\right)$	
$\frac{10}{9}$	$\left(0, \pm\sqrt{\frac{2}{5}}\right)$	$-\frac{1}{270}\left(\frac{\partial^6 f}{\partial\xi^4\,\partial\eta^2}+\frac{\partial^6 f}{\partial\xi^2\,\partial\eta^4}\right)$
$-\frac{8}{9}$	$(0, 0)$	

$n = 12$

w_i	(ξ_i, η_i)	E
$\frac{16}{225}$	$(\pm 1, \pm 1)$	
$\frac{8}{45}$	$(0, \pm 1)$	$-\frac{1}{6825}\left(\frac{\partial^6 f}{\partial\xi^6}+\frac{\partial^6 f}{\partial\xi^6}\right)$
$\frac{8}{45}$	$(\pm 1, 0)$	
$\frac{169}{225}$	$\left(\pm\sqrt{\frac{3}{13}}, \pm\sqrt{\frac{3}{13}}\right)$	$-\frac{2}{1755}\left(\frac{\partial^6 f}{\partial\xi^4\,\partial\eta^2}+\frac{\partial^6 f}{\partial\xi^2\,\partial\eta^4}\right)$

Table I. (*continued*)

$n = 13$

w_i	(ξ_i, η_i)	E
$\frac{1}{25}$	$(\pm 1, \pm 1)$	$-\frac{37}{88905600}\left(\frac{\partial^8 f}{\partial \xi^8}+\frac{\partial^8 f}{\partial \eta^8}\right)$
$\frac{98}{405}$	$\left(\pm\sqrt{\frac{6}{7}}, 0\right)$	
$\frac{98}{405}$	$\left(0, \pm\sqrt{\frac{6}{7}}\right)$	$-\frac{1}{151200}\left(\frac{\partial^8 f}{\partial \xi^2\,\partial \eta^6}+\frac{\partial^8 f}{\partial \xi^6\,\partial \eta^2}\right)$
$\frac{1024}{2025}$	$\left(\pm\sqrt{\frac{3}{8}}, \pm\sqrt{\frac{3}{8}}\right)$	
$\frac{344}{405}$	$(0, 0)$	$-\frac{1}{14400}\frac{\partial^8 f}{\partial \xi^4\,\partial \eta^4}$

$n = 16$

w_i	(ξ_i, η_i)	E
$-\frac{103}{1350}$	$(\pm 1, \pm 1)$	$\frac{3497}{14033250}\left[\frac{\partial^6 f}{\partial \xi^6}+\frac{\partial^6 f}{\partial \eta^6}\right]$
$\frac{9}{100}$	$\left(\pm\frac{1}{3}, \pm 1\right)$	
$\frac{9}{100}$	$\left(\pm 1, \pm\frac{1}{3}\right)$	
$\frac{121}{135}$	$\left\{\pm\sqrt{\frac{5}{11}}, \pm\sqrt{\frac{5}{11}}\right)$	$+\frac{59}{14850}\left[\frac{\partial^6 f}{\partial \xi^4\,\partial \eta^2}+\frac{\partial^6 f}{\partial \xi^2\,\partial \eta^4}\right]$

$n = 16$ (Lobatto Integration)

w_i	(ξ_i, η_i)	E
$\frac{1}{36}$	$(\pm 1, \pm 1)$	$\frac{-4}{23625}\left[\frac{\partial^6 f}{\partial \xi^6}+\frac{\partial^6 f}{\partial \eta^6}\right]$
$\frac{5}{36}$	$\left(\pm\sqrt{\frac{1}{5}}, \pm 1\right)$	
$\frac{5}{36}$	$\left(\pm 1, \pm\sqrt{\frac{1}{5}}\right)$	
$\frac{25}{36}$	$\left(\pm\sqrt{\frac{1}{5}}, \pm\sqrt{\frac{1}{5}}\right)$	

Table I. (*continued*)

$n=17$		
w_i	(ξ_i, η_i)	E
$\frac{-191}{11280}$	$(\pm 1, \pm 1)$	
$\frac{9}{80}$	$\left(\pm\frac{1}{3}, \pm 1\right)$	$\frac{752}{399735}\left[\frac{\partial^6 f}{\partial\xi^4\,\partial\eta^2}+\frac{\partial^6 f}{\partial\xi^2\,\partial\eta^4}\right]$
$\frac{9}{80}$	$\left(\pm 1, \pm\frac{1}{3}\right)$	
$\frac{586971}{1191920}$	$\left(\pm\sqrt{\frac{317}{693}}, \pm\sqrt{\frac{317}{693}}\right)$	
$\frac{5696}{4755}$	$(0, 0)$	

accuracy. However because many of the integration points coincide with nodal locations, the overall computer work required is reduced in some cases. Gray[7] and Young[8] have previously exploited this type of computational savings.

As an illustrative example, consider a two-dimensional quadratic isoparametric element which has nine nodes and uses Lagrangian basis functions. This element can be transformed into a square centred at $(\xi, \eta)=(0, 0)$ with corners at $(\xi, \eta)=(\pm 1, \pm 1)$ through the Jacobian, J. Simpson's nine point rule and 2×2 GQ provide the same order of accuracy for integration over this transformed square region. To compute the Jacobian of the transformation from global to local co-ordinates, it is necessary to evaluate, at each of the integration points:

$$\frac{\partial x}{\partial \xi}=\sum_{i=1}^{9} x_i \frac{\partial \phi_i}{\partial \xi} \tag{1a}$$

$$\frac{\partial x}{\partial \eta}=\sum_{i=1}^{9} x_i \frac{\partial \phi_i}{\partial \eta} \tag{1b}$$

$$\frac{\partial y}{\partial \xi}=\sum_{i=1}^{9} y_i \frac{\partial \phi_i}{\partial \xi} \tag{1c}$$

$$\frac{\partial y}{\partial \eta}=\sum_{i=1}^{9} y_i \frac{\partial \phi_i}{\partial \eta} \tag{1d}$$

where x_i, y_i are the co-ordinates of node i in the global x–y system
ξ, η are the local co-ordinates
$\phi_i(\xi, \eta)$ is the basis function associated with node i.

Using 2×2 GQ over the element, each of these four functions must be evaluated at the four quadrature points using a sum which requires nine multiplications. Thus 144 multiplications must be performed. Additionally eight more multiplications are performed to get the Jacobian at the four locations from equations (1) for a total of 152. For Simpson's rule, the four functions in (1) must be evaluated at nine points. However because first derivatives of each basis function are non-zero at only three nodal locations or quadrature points, only three multiplications need be performed to evaluate a series in (1) at a node. Thus only 108 multiplications need be performed to obtain $\partial x/\partial\xi$, $\partial x/\partial\eta$, $\partial y/\partial\xi$, and $\partial y/\partial\eta$. To complete the determination of the Jacobian at the required locations, 18 additional multiplications must be performed for a total of 126. Thus Simpson's rule provides a 17 per cent saving in multiplications needed for computation of the Jacobian when compared to 2×2 GQ when quadratic Lagrangian basis functions are used.

To continue with this example, evaluate the integral

$$\int_{-1}^{1}\int_{-1}^{1} \phi_i\phi_j J\,\mathrm{d}\xi\,\mathrm{d}\eta \qquad \begin{matrix} i=\overline{1,9} \\ j=\overline{1,9} \end{matrix} \tag{2}$$

for all i and j. Because the integrand is symmetric with respect to i and j, only 45 different integrals need to be evaluated. Each integral can be evaluated using 2×2 GQ by performing eight multiplications. Thus 360 multiplications are necessary to evaluate the set of integrals in (2). On the other hand, only 9 total multiplications are necessary using Simpson's rule. This significant reduction in work is achieved because for quadratic Lagrangian basis functions, $\phi_i=0$ at all nodes (Simpson's rule integration points) except at node i. Thus

$$\int_{-1}^{1}\int_{-1}^{1} \phi_i\phi_j J\,\mathrm{d}\xi\,\mathrm{d}\eta \simeq \begin{cases} 0 & i\neq j \\ c_iJ_i & i=j \end{cases} \tag{3}$$

where J_i is the Jacobian evaluated at node i and $c_i=\frac{1}{9}$, $\frac{4}{9}$, or $\frac{16}{9}$ depending on whether i is a corner, midside, or mid-element node respectively.

Additionally if terms computed from equation (2) yield coefficients which are inserted into a matrix, Simpson's rule will give no off-diagonal terms while GQ will. Thus significant savings may be realized in the subsequent solution of the matrix problem.

If we evaluate the integral

$$\int_{-1}^{1}\int_{-1}^{1} \frac{\partial\phi_i}{\partial x}\frac{\partial\phi_j}{\partial x} J\,\mathrm{d}\xi\,\mathrm{d}\eta$$

by 2×2 Gauss quadrature and Simpson's rule, the savings will not be as great as in the previous case although significant savings will still be realized. However, the resulting 9×9 element matrix will be dense using 2×2 GQ while Simpson's rule generates 36 zero elements. Thus the global matrix formed will be more sparse using Simpson's rule than GQ and the system of equations can be solved more rapidly.

This comparison admittedly is made using Lagrangian quadratic basis functions which strongly favour non-Gaussian quadrature. Such dramatic savings will not be achieved using some other type of basis function. However the above example seems to illustrate that a judicious choice of integration formula depending on the type of element and basis functions being used can greatly enhance computational efficiency. The coincidence of nodes and quadrature points is a key factor in obtaining such an efficient scheme. A number of two-dimensional integration formulas, along with their truncation errors, are presented in Table I. For completeness and to facilitate a comparison between error terms, GQ and Lobatto formulas are included in the table along with formulas not previously in the literature. The selection of any of these alternative formulas depends, as mentioned above, on the element type and the basis functions chosen, and hence will ultimately be determined by the judgement and programming skill of the prospective user. For instance, by defining cubic isoparametric basis functions such that each node coincides with a Lobatto integration point, the 16-point

Lobatto formula may compete well with the 9-point GQ. Alternatively when the nodes in the cubic element are equispaced, the other listed 16-point formula or the 17-point formula may prove to be the most useful.

Finally, extension of the listed formulas to three-dimensions is a straightforward procedure. Computational savings over three dimensional GQ are expected to be substantial.

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