Soil Water Modeling II: On Sensitivity to Finite Difference Grid Spacing

C. R. Amerman, E. J. Monke
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ABSTRACT

THE influence of finite difference grid size on soil water flow model accuracy was studied using two cases. One case was of steady, two-dimensional infiltration using the successive overrelaxation (SOR) method of solving finite difference equations. The other was of transient, two-dimensional infiltration using the alternating direction implicit (ADI) method. We formulated finite difference expressions for both cases using central differencing techniques. In the transient case, a very small grid size in both time and space dimensions was necessary at the initiation of infiltration. Furthermore, the ADI method for this nonlinear case was only conditionally stable—at least at the initiation of infiltration. As infiltration proceeded, however, both the time and space grid sizes could be made larger. Indicators of a grid size that was obviously too coarse were a fluctuating infiltration rate in the transient case and irregularly shaped equipotential lines in the steady-state case.

Although the models were nonlinear, they both converged; i.e., successively smaller grid sizes yielded solutions that asymptotically approached a limit. For a given regular grid size, considerable computational saving could be effected without appreciable loss of accuracy by using an irregular grid in which the regular grid size was duplicated in the part of the section exhibiting the greatest curvature of equipotential lines, while larger grid sizes could be used in other parts of the section. Smaller grid sizes were also needed in regions where hydraulic gradient changed rapidly.

Accuracy of estimation varied approximately with the inverse of grid size rather than with the square of the inverse, as is generally claimed for central differencing on a square grid.

INTRODUCTION

One method for investigating soil water movement is to approximate the partial differential equations for porous media flow with finite difference models (Hanks and Bowers, 1962; Reisenauer et al., 1963; Taylor and Luthin, 1963, 1969; Freeze, 1971; Amerman, 1976a, 1976b). A measure of the degree of approximation is the discretization error, U-u, where U is the exact solution of the partial differential equation, and u is the exact solution of the finite difference equation (Smith, 1965). Forsythe and Wasow (1960) discussed discretization error and gave some appraisal techniques. However, these techniques have certain weaknesses even when applied to the linear Dirichlet boundary problem (all boundary conditions given in terms of U) with smooth boundary data. Vitasek (1969) claimed that these methods yield inaccurate estimates of error.

Equations for unsaturated soil water movement are nonlinear. Furthermore, boundary conditions are usually of the mixed type (both Dirichlet- and Neumann-type, where the latter refers to boundary conditions given as the gradient of U normal to the boundary), and boundary data are usually not smooth (boundary geometry may be complex and may contain junctions between Dirichlet and Neumann types). Considering the weaknesses and lack of accuracy claimed for the methods discussed by Forsythe and Wasow (1960), when applied to ideal situations, we concluded that these methods would be of little value with finite difference models of soil water flow.

A gross approximation to discretization error is afforded by the order of accuracy to which a finite difference equation represents a partial differential equation, where order of accuracy is expressed as a function of grid spacing (Δx and Δy). Vitasek (1969) gave a table of orders of accuracy for several finite difference formulations. From this table, we observed that order of accuracy and, therefore, discretization error, is dependent upon grid spacing, arrangement of nodes in the finite difference grid (square, rectangular, triangular, etc.) and the manner in which finite difference expressions are formulated (number and pattern of the nodes included in the expressions). Order of accuracy in a rectangular grid is influenced by whether the grid is regular (Δx and Δy uniform throughout a given section) or irregular (Δx and Δy variable over the section).

The lack of rigorous methods for evaluating discretization error, especially for nonlinear applications, has apparently discouraged those who model porous media flow from considering model accuracy.

Exceptions are Taylor and Luthin (1963) and Smith and Woolhiser (1971). Taylor and Luthin discussed the effect of grid spacing upon model accuracy for saturated flow, a condition for which the flow equations are linear. Boundary conditions were of mixed type, however, and the methods of Forsythe and Wasow (1960) could not be confidently applied. Instead, Taylor and Luthin (1963) used a simple, direct technique of running their model several times for the same prototype, but using a different grid spacing for each run. Then, they plotted certain model output parameters as a function of grid size. The resultant graphs provided a visual test of model sensitivity to grid size. Smith and Woolhiser (1971) used the same technique with the problem of transient infiltration into unsaturated soil.

The purpose of this paper is to discuss the sensitivity

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The authors are: C. R. AMERMAN, Research Hydraulic Engineer, ARS, USDA, Columbia, MO; E. J. MONKE, Professor, Agricultural Engineering Dept., Purdue University, West Lafayette, IN.

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to grid spacing of two finite difference models of porous media flow. Scale effects and the sensitivity of one model to uniformity of grid spacing were also investigated. Sensitivity testing methods were similar to those of Taylor and Luthin (1963).

PARTIAL DIFFERENTIAL EQUATIONS

Water moves through porous materials in response to gradients in hydraulic head and in accordance with the law of continuity. Due to the three-dimensional nature of the distribution of hydraulic head and of hydraulic soil properties, partial differential equations together with appropriate boundary and initial conditions are used to specify particular flow situations. However, three-dimensional analysis of porous media flow is generally not economically feasible. A two-dimensional version of Richards' equation for porous media flow was given by Childs (1969):

$$\frac{\partial \Theta}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial H}{\partial y}\right) \quad [1]$$

Θ = volumetric water content
t = time (T)
H = h + z = hydraulic head (L)
h = h(Θ) = soil water pressure head (L)
z = elevation above a datum (L)
K = K(h) = soil hydraulic conductivity (LT^{-1})
x, y are distances along the horizontal and vertical axes, respectively, of the Cartesian coordinate system (L).

For computational convenience, we use the h-based form of Richards' equation, given for general orientation of the Cartesian coordinate axes:

$$C \frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K \frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial h}{\partial y}\right) + \sin \alpha \frac{\partial K}{\partial x} + \cos \alpha \frac{\partial K}{\partial y} \quad [2]$$

where
C = $\partial \Theta / \partial h$ = specific water capacity (L^{-1})
α = angle of rotation of the coordinate axes.

For steady flow, the left side of equation [2] is zero, and boundary conditions (but not initial conditions) completely specify a problem.

Equation [2] is a highly nonlinear equation, but experience and tests like those of Taylor and Luthin (1963) have shown that successive overrelaxation (SOR) and alternating direction implicit (ADI) finite difference models generally converge to a reasonable approximation to the solution when iteration is used to remove the nonlinearity.

A steady-state model employed the SOR (Smith, 1965; Forsythe and Wasow, 1960) technique to solve finite difference equations which simulated steady flow for either saturated or unsaturated conditions or for a combination of the two (water table condition). A finite difference transient model used the ADI (Peaceman and Rachford, 1955) method of solution and simulated unsteady flow for unsaturated conditions. The SOR and ADI models were similar because finite difference expressions for both were formulated using central differencing on a square or rectangular grid. Thus, of the variables which influence accuracy, only grid spacing and uniformity of grid spacing were allowed to vary.

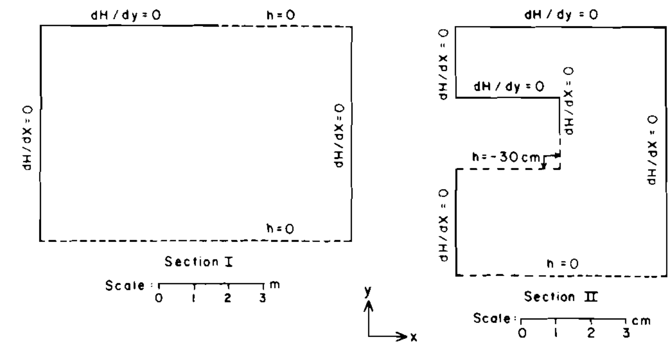

FIG. 1 Sections and boundary conditions used in sensitivity tests.

TEST SECTIONS

Two sections, with no particular physical significance, were used to test the sensitivity of the SOR model. These sections and their boundary conditions are shown in Fig. 1. A section similar to Section I, but of slightly different proportions, was used in a sensitivity test of the ADI model.

The datum from which the elevation z was measured was the bottom boundary (water table) in each case.

The grid spacings used in testing Section I are given below in which the equality Δx, Δy = a, b, c means that both Δx and Δy take on the values a, b, c in different parts of the grid:

A. Regular Grid
   1. $\Delta x = \Delta y = 1$ m
   2. $\Delta x = \Delta y = 0.5$ m
   3. $\Delta x = \Delta y = 0.25$ m
   4. $\Delta x = \Delta y = 0.125$ m
B. Irregular Grid
   1. Δx, Δy = 0.125, 0.25, 0.5 m
   2. Δx, Δy = 0.0625, 0.125, 0.25 m

Grid spacings in Section II were:

A. Regular Grid
   1. $\Delta x = \Delta y = 1$ cm
   2. $\Delta x = \Delta y = 0.5$ cm
B. Irregular Grid
   1. Δx, Δy = 0.5, 1.0 cm

For fully saturated flow in homogeneous, isotropic media, K has the same value in all parts of the cross section. Equation [2] reduces to the linear Laplace equation, and the constant, K, is eliminated. Solutions to the Laplace equation are independent of scale, i.e., a solution may be obtained for a particular geometry and applied to any flow region of the same geometrical proportions, regardless of actual size.

In unsaturated flow situations, however, K varies approximately exponentially with water content and may exhibit large differences over a flow section. For this reason, one cannot model a large section with a geometrically similar smaller one without making scaling adjustments in the relationship between K and h. Because of the exponential relationship, this is a complicated task. The question naturally arises whether there is an advantage to modeling a large section with a geometrically similar one for unsaturated flow. An advantage might be realized if modeling with a smaller section meant that fewer nodes were needed in the solution grid so that fewer computations would be necessary.

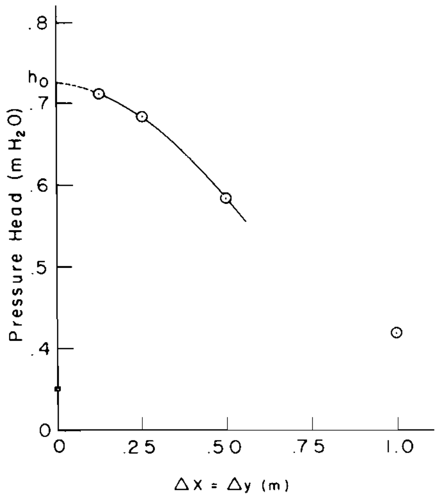

FIG. 2 Example of estimation of pressure head for $\Delta x = \Delta y = 0$ [h_0] for Section I, steady flow.

Hydraulic conductivity as a function of water content varies widely between soils (Bouwer, 1964). For simulation of flow in Section I, values of K(h) and C(h) were calculated from data given by Hanks and Bowers (1962) for Sarpy Loam soil. Amerman (1976a) gave K(h) for the Saybrook Silt Loam soil used in simulations involving Section II. Saturated hydraulic conductivity for Saybrook S. L. was nearly two orders of magnitude higher than that for Sarpy L. and the latter decreased much more rapidly than the former as pressure head decreased, although the curves were very similar in shape. In each case, the simulation used the K(h) data in tabular form.

Differences between Sections I and II, then, include geometry, scaling, and the hydraulic conductivity-pressure head relationship.

SENSITIVITY OF A STEADY-STATE SOR MODEL

In earlier papers Amerman (1976a, 1976b) discussed a steady-state porous media flow model based on the SOR method of solving finite difference equations. The uniqueness of the model lies, not in the solution technique, but in the mechanisms by which various geometries and combinations of boundary conditions can be accommodated. Therefore, the sensitivity data we present here should apply in principle to any nonlinear SOR model using central differences on a rectangular grid. Differences in the effect of the nonlinear coefficient, K(h) in this case, were not extensively explored, however.

Model output consisted of values of h and of H at each node of the finite difference grid. For the regular grids of Section I, the spacing of the coarse grid ($\Delta x = \Delta y = 1$ m) was an even multiple of each of the finer spacings; thus, each node in the coarse grid had a spatially identical counterpart in each of the finer grids. To make the largest possible number of four-way comparisons over the range of Δx and Δy, we plotted h as a function of grid spacing for each node of the coarsest grid. Fig. 2 shows such a plotting for a representative node. At each of these nodes, the plotted points for the three finer grid spacings were connected by a simple (without inflection), monotonic curve. Furthermore, almost all of the curves approached zero slope as they neared $\Delta x = \Delta y = 0$. Exceptions seemed to be related to the singular point at the junction between the infiltrating and impermeable parts of the upper surface. This is evidence that the SOR model was indeed converging. Extrapolating the curves to the h-axis gave estimates of h (h_0) at $\Delta x = \Delta y = 0$.

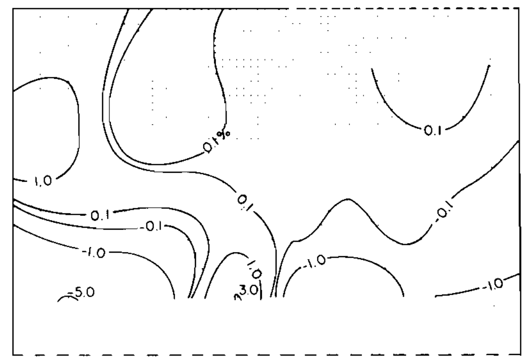

FIG. 3 Distribution of differences in h between regular grid spacing of 0.125 m and irregular grid spacings of 0.125, 0.25, and 0.5 m [dots show nodes of irregular grid], Section I, steady flow.

Improvement of estimation of h or H by decreasing the grid spacing is accomplished at the expense of computation time, because the number of computational operations on a two-dimensional grid is inversely proportional to the square of grid spacing. An irregular grid may provide a means of compromising between cost and accuracy.

Part of the upper surface boundary of Section I is impermeable, and the remainder is saturated. The greatest curvature in equipotentials will be expected in the neighborhood of the junction between the two types of boundaries. This region of strong curvature should probably be covered with a fine grid, and coarser grids might then suffice for areas of lesser curvature.

The dots on Fig. 3 show the locations of grid nodes for the irregular grid in which Δx and Δy took the values 0.125, 0.25, or 0.5 m. The curves on Fig. 3 are lines of equal percent difference between h-values obtained with the irregular grid and h-values obtained with a uniform grid spacing of 0.125 m. The curves do not continue to the infiltrating and water table boundaries, because h is known exactly on these boundaries.

When central differencing is used, Vitasek (1969) gives an order of accuracy of $0((\Delta x)^2)$ for a regular grid and $0(\Delta x)$ for an irregular grid. However, in our case, the irregular grid yielded an h-distribution very close (<5 percent difference) to that obtained by using a uniform grid with the same spacing as the finest part of the irregular grid.

The sensitivity of the steady-state SOR finite difference model to grid spacing, as applied to Section I, is summarized in Table 1. The extrapolated solution re-

TABLE 1. STATISTICS OF COMPARISONS (PERCENTAGE DIFFERENCES) OF SOLUTIONS OF MODEL AT VARIOUS GRID SPACINGS TO EXTRAPOLATED SOLUTION WITH ZERO GRID SPACING.

Grid spacing, m	Mean, m	Std. dev., m	Max., m	Min., m
Regular				
1	30.7	13.0	63.2	10.1
0.5	14.7	7.2	38.4	0.7
0.25	5.0	3.3	17.6	0
0.125	2.1	1.9	9.2	0.1
Irregular				
0.125, 0.25, 0.5	2.2	2.0	9.2	0
0.0625, 0.125, 0.25	1.3	1.2	6.6	0

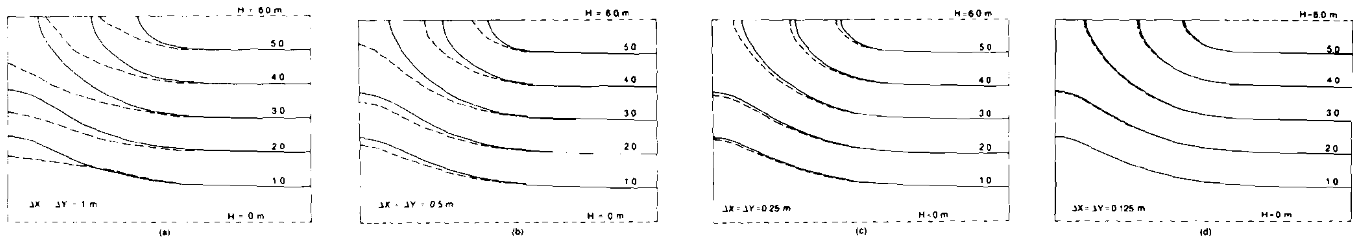

FIG. 4 Influence of grid spacing on equipotential lines for Section I, steady flow. Solid black curves are for the irregular grid of 0.0625-, 0.125-, and 0.25-m spacing. Spacings for dashed curves for the regular grids are given under each part of the figure.

ferred to in the title of Table 1 was the h_0- array obtained on the 1-m grid spacing from plots of the type portrayed by Fig. 2. The trend of values in Table 1 supports the claim, made for Fig. 2, that as grid size decreases, the solution converges toward the solution of the soil water flow equation.

Fig. 4 shows how the effect of grid spacing was distributed over the section. The solid curves are equipotentials (lines of equal hydraulic head, H) obtained using the irregular grid, where Δx and Δy took the values 0.0625, 0.125, and 0.25 m. As indicated in Table 1, this grid probably most closely approximates the solution of the differential equation. The dashed curves represent the equipotentials for the coarser, regular grids, as indicated in the label for each part of the figure.

Decreasing the grid spacing from 1 to 0.5 m produced the most striking improvement in the equipotentials. The 1-m grid spacing gives irregularly shaped equipotentials. An irregular distribution of h or of H is a good indication that the grid spacing is too large to yield even a reasonable estimate of the h distribution. We also noticed in the several parts of Fig. 4 that sensitivity to grid size is apparently related to two-dimensionality of flow. On the right-hand sides of Fig. 4, where flow direction was essentially vertically downward, the curves obtained with the different grid intervals coincided. As the curves began to curve upward on the left side of each part of the figure, departures began to be apparent.

Fig. 5 shows equipotentials for Section II. The dashed-line curves were obtained using the largest regular grid (1 cm), and the solid-line curve resulted from the 0.5-cm regular grid. The large dots represent the results obtained using the irregular grid of 0.5 and 1 cm. Fig. 6 shows isolines of percent difference in the results between the irregular grid of 0.5- and 1-cm grid spacings and the regular grid of 0.5-cm spacing.

Although Section II is quite different from Section I in shape, scale, and boundary conditions, the observations made with regard to Section I are essentially confirmed in Figs. 5 and 6. The 1-cm regular grid, being too coarse, yielded irregular equipotentials. The 0.5-cm regular grid was an improvement, because the equipotential curves were smoother. The irregular grid of 0.5 cm and 1 cm produced nearly the same results as the 0.5-cm regular grid.

In Fig. 5 sensitivity to grid size is much more apparent in the zone of nearly one-dimensional flow below the notch than in the one-dimensional, right-hand parts of Fig. 4. We also noticed that the vertical gradient $(\partial H/\partial y)$ in Fig. 5 increases with distance up the left-hand boundary. This is a reflection of the fact that soil water content decreases along the path from the lower boundary ($h = 0$ cm) to the bottom of the notch ($h = -30$ cm). Thus, hydraulic conductivity decreases in the same direction. Because the volume rate of flow must remain constant for steady flow, Darcy's law demonstrates that the gradient must increase when one-dimensional flow follows a path from wet to dry soil. In Fig. 4, soil water content is nearly constant down the right-hand side (top and bottom boundary conditions are the same), and

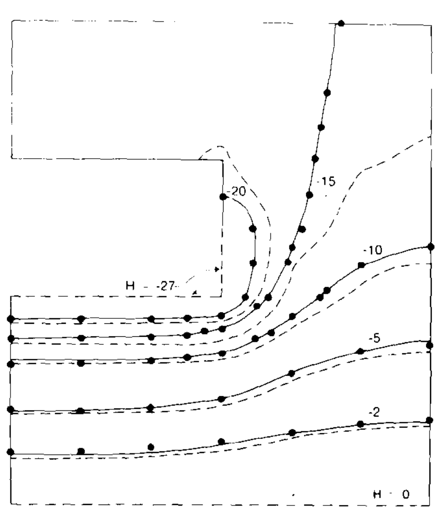

FIG. 5 Influence of grid spacing on equipotential lines for Section II, steady flow. Solid black curves are for 0.5-cm regular grid, dashed curves are for 1-cm regular grid, and solid dots resulted from an irregular grid of 0.5 and 1 cm.

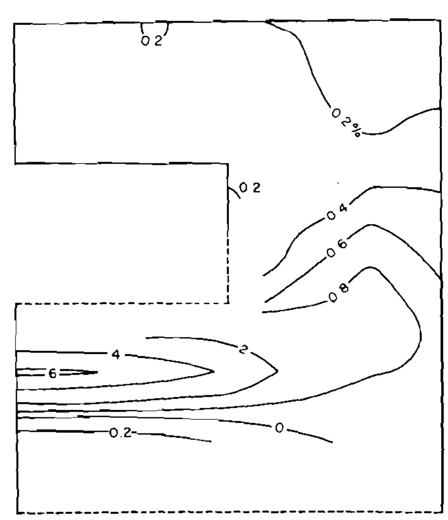

FIG. 6 Distribution of percent difference in h between regular grid spacing of 0.5 cm and irregular grid spacings of 0.5 and 1 cm [dots show nodes of irregular grid], Section II, steady flow.

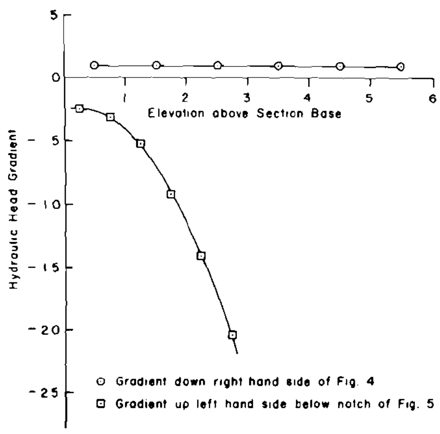

FIG. 7 Comparison of hydraulic gradient distributions in one-dimensional flow portions of Sections I and II, steady flow.

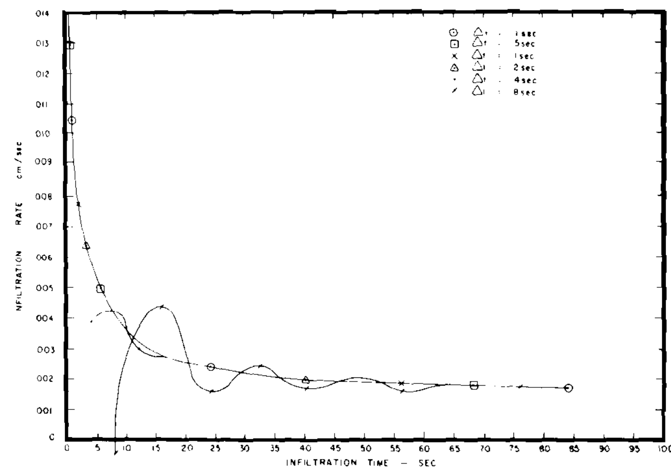

FIG. 8 Effect of time increments Δt on estimating infiltration rate/time curve, Section I.

the gradient is nearly uniform with depth as indicated by essentially equal spacing of the equipotentials. Fig. 7 illustrates these observations. The circled points show that the gradient down the right-hand side of Section I is essentially constant, with the value one. This is the gravity component of gradient; therefore, the gradient in the soil water pressure head is zero. The squared points show the gradient up the left side of Section II between the water table and the bottom of the notch and show that there is considerable gradient in soil water pressure head.

We conclude that sensitivity to grid spacing is influenced both by curvature of equipotentials (two dimensionality of flow) and by variation in the magnitude of the gradient.

SENSITIVITY OF A TRANSIENT ADI MODEL

Sensitivity of a transient finite difference model to spatial and temporal grid size is a much more complex problem than one involving only steady flows. The data we present here are not extensive, but they illustrate some interesting possibilities for consideration by those using finite difference models of transient phenomena. Further numerical experimentation is needed for more complete understanding of the relationships between the several factors that influence the accuracy and stability of finite difference approximations to the solution of equation [2]. For transient problems, in particular, differences in K(h) and C(h) relationships may affect sensitivity to mesh size.

The ADI method of solution (Peaceman and Rachford, 1955) was used in constructing a model for two-dimensional, transient, unsaturated porous media flow (Amerman, 1969). A section and boundary conditions similar to Section I, Fig. 1, but of slightly different proportions, were used in a sensitivity test of that model using regular, square grids.

The initial condition was that of drainage to equilibrium. At time zero, the dashed portion of the upper boundary of Section I was exposed to zero pressure head. It became an infiltrating boundary, and flow discharged across the lower boundary.

Solution of equation [2] yielded data from which we plotted instantaneous equipotential diagrams, like those in Fig. 4, at the end of each time step. Plotting a few equipotentials from solutions, using different values of Δx and Δy while holding Δt constant, indicated essentially the same phenomenon as the comparisons of Figs. 4 and 5. Large values of Δx and Δy (i.e., coarse grids) resulted in irregular equipotential lines.

The influence of Δt may be illustrated by considering the variation of infiltration rate with time, as in Fig. 8. For this comparison, Δx and Δy were held constant at 1 cm. A fluctuating or irregularly shaped infiltration curve indicated a time interval that was much too coarse. However, the fluctuations damped out, so that even with excessively large Δt, a nearly correct solution resulted after infiltration had proceeded for a long time.

A finite difference model is a discrete approximation to a continuous phenomenon. Equation [2] obeys the law of continuity, but the finite difference model only approximates continuity. Fig. 9 shows that, for $\Delta x = \Delta y = 0.5$ cm, when approximate equilibrium is reached, outflow exceeds inflow. The x marks at time = 180 sec show the equilibrium separation between inflow and outflow with $\Delta x = \Delta y = 1$ cm. Since the agreement between inflow and outflow is better for the smaller grid interval, we conclude that decreasing Δx and Δy gives a better approximation to continuity.

Fig. 10 is rather complicated, but when studied carefully it shows how time and spatial increment sizes interacted to affect infiltration rate. An individual curve on this figure is a plot of infiltration rate versus time increment size for a given grid increment size and at a given

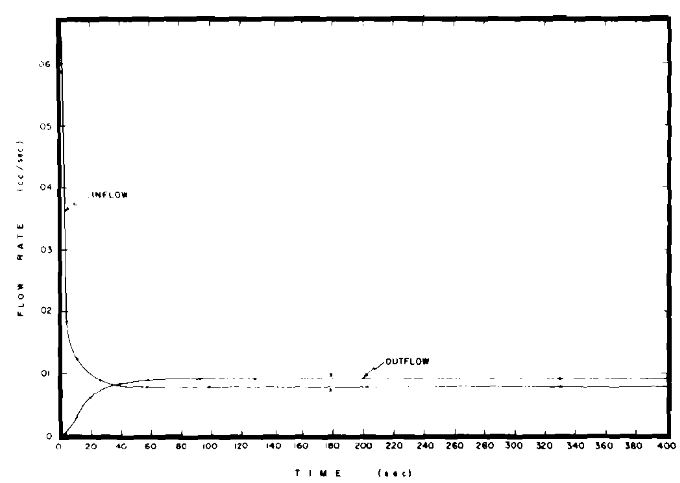

FIG. 9 Effect of grid interval, Δx, upon obedience of finite difference model to law of continuity. Curves are for $\Delta x = \Delta y = 0.5$ cm. Two x marks at 180 sec show positions of equilibrium curves for $\Delta x = \Delta y = 1$ cm.

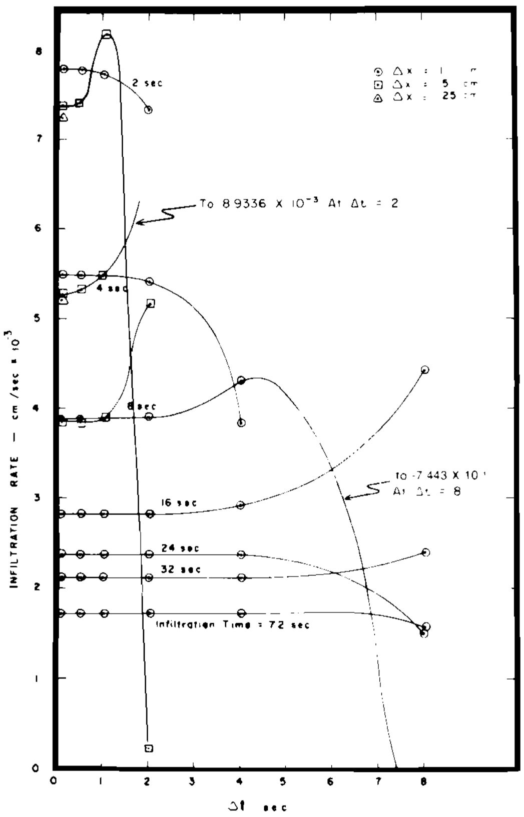

FIG. 10 Effect of interaction of spatial and temporal grid spacings on estimated infiltration rate at selected infiltration times, Section I.

infiltration time, i.e. the time elapsed since infiltration began. To illustrate, all of the data points in the group of curves that intersect the ordinate axis between infiltration rates of 7 x 10^{-3} and 8 x 10^{-3} cm/sec were obtained after water had infiltrated for 2 sec. Those in the next grouping down are estimates of infiltration rate at the end of 4 sec and so on.

Within a group of curves for a given infiltration time, Δx varies between curves ($\Delta x = \Delta y$). In scanning down the ordinate axis we observed that within groups at successively longer infiltration times, the curves for different Δx intersected the ordinate axis closer to each other. Finally, after about 8 sec of infiltration, differences within the size range of Δx tested did not affect infiltration rate. We conclude, therefore, that one may be able to increase simulation efficiency by allowing Δx to increase with simulation time.

An individual curve on Fig. 10 shows how, for a given Δx and a selected infiltration time, the solution converged to an estimate of a true solution as Δt tended toward zero. After 2 sec of infiltration, varying Δt with $\Delta x = 1$ cm yielded a smooth, monotonic curve which approached the horizontal at its intersection with the ordinate axis. Holding Δx constant at 0.5 cm yielded a fluctuating curve for which the fluctuations grew with increasing Δt. With constant $\Delta x = 0.25$ cm and varying $\Delta t > 0.1$ sec, the fluctuations also grew as Δt increased, but in this case were so violent that even the estimated infiltration rate for the second Δt value could not be plotted at the scale of Fig. 10. This reaction satisfies the definition of instability.

Richtmyer and Morton (1967) give the necessary stability condition for solving finite difference equations by explicit methods as:

$$\frac{\Delta t}{(\Delta x)^2} < \frac{1}{2}\sigma \quad \text{[1]}$$

where σ is a parameter which depends on equation coefficients (K in the case of equation [2]). For $\Delta x < 1$, then, larger Δx values are less likely to lead to instability for a given series of Δt values. The results shown in Fig. 10 are not inconsistent with equation [3].

The ADI method is an implicit technique for solving finite difference equations and Richtmyer and Morton (1967) classify it as unconditionally stable. However, they discuss it only in the context of its application to linear equations. We conclude that the observed instability for small Δx and Δt in the first few seconds of infiltration simulation is an artifact of adapting a linear implicit method of solution to a nonlinear finite difference equation.

If compatible values of Δx and Δt are used at the beginning of infiltration, Fig. 10 also shows that as infiltration time advances (as one compares curves from top to bottom of the figure), larger and larger values of Δt give essentially the same estimate of infiltration rate as values very near zero. Thus, except for certain restrictions which apparently apply near the beginning of a solution run, ADI applied to a nonlinear finite difference equation obeys the well-known rule for implicit schemes that Δt can be increased as solution time advances.

Further numerical experiment might show how both Δx and Δt can be increased with solution time so as to obtain a simulation scheme with optimum efficiency.

ORDER OF ACCURACY

We noted earlier that a regular grid should yield an accuracy of $0((\Delta x)^2)$. If Δx is halved, for example, the discretization error should be quartered. Inspection of Table 2, developed using SOR data from the steady-state case, shows that the error varied more nearly with Δx rather than $(\Delta x)^2$, at least for the coarser grids. A regular grid spacing of 0.5 m resulted in an estimated error that was about half of that of the 1-m grid size. Also, the error for the 0.25-m regular grid was about one-sixth of that for the 1-m grid size as compared with the one-sixteenth expected. Table 2 shows comparisons for all grids supplied to Section I using the SOR model. Comparisons for the small grid sizes may not be very useful, however, because h_0 is estimated by visual curve fitting and extrapolation. The magnitude of the unknown error between estimated and true h_0 may approach the magnitude of the error between h for a small grid spacing and estimated h_0.

Possible reasons for the discrepancies between the second and fourth columns of Table 2 include nonlinearity of the flow equation and the use of Neumann boundary conditions. Finite difference theory for nonlinear systems is much less comprehensive than for linear systems (Greenspan, 1965) but experience has shown that linear finite difference methods often can be successfully used for nonlinear problems. However, certain details of the methods, like the influence of the overrelaxation factor in the SOR method, do not conform with linear theory and experience (Amerman, 1976b). Prediction of order of accuracy, based on grid intervals, may be another nonconforming detail.

Textbooks discuss order of accuracy in the context

TABLE 2. COMPARISON OF PREDICTED AND ESTIMATED OBSERVED ORDERS OF ACCURACY USING SOR*.

Grid spacing (Δx)	Predicted $[(\Delta x)^2]$	Mean percent error	Observed (mean/30.7)
Regular			
1	1	30.7	1
0.5	0.25	14.7	0.48
0.25	0.0625	5.0	0.16
0.125	0.0156	2.1	0.068
Irregular			
0.125, 0.25, 0.5	0.0156	2.2	0.072
0.0625, 0.125, 0.25	0.0039	1.3	0.042

*SOR—The successive overrelaxation technique of finite differencing.

of grid shape and spacing and finite difference technique but not of boundary condition. Recall that a Neumann-type boundary condition is one for which the gradient of the dependent variable is normal to the boundary. In a finite difference scheme, this gradient must be specified by a finite difference expression and, therefore, constitutes a level of approximation over and above that attributable to the finite difference scheme (Greenspan, 1973). This approximation may contribute to decreasing the accuracy of a given grid and finite difference technique.

Textbooks and reference works (Forsythe and Wasow, 1960; Smith, 1965; Vitasek, 1969) discuss order of accuracy in the context of an apparently dimensioned Δx. The scales of Sections I and II (Fig. 1) differ by two orders of magnitude. Yet, the number of spatial intervals into which a unit length of each section must be divided to achieve reasonably accurate solutions is essentially the same. This observation extends to the use of varying grid sizes in an irregular mesh and holds even for dissimilar geometry. Specifically, the visual smoothness characteristics are similar between the equipotential curves of Section I, for which Δx and Δy were 1 and 0.5 m, and the curves of Section II, for which Δx and Δy were 1 and 0.5 cm.

We conclude that there is no computatinal advantage gained by modeling a large cross section with a geometrically similar smaller one; when using a finite difference technique both require about the same number of nodes.

Thus, when curvature of equipotentials and hydraulic gradients are accounted for, the sizes of Δx and Δy may be determined, not as specific lengths, but as proportions of the unit length of the cross section in the horizontal and vertical directions, respectively.

SUMMARY AND CONCLUSIONS

Two nonlinear, two-dimensional, finite difference models of soil water flow—one a steady-state simulation using successive overrelaxation methods, the other a transient model using alternating direction implicit techniques—were investigated for sensitivity to grid spacing. In each case, finite difference formulation was by central differencing on a square or rectangular grid.

There is no rigorous method for determination of discretization error for nonlinear finite differencing. The investigation was empirical, using data produced by several model runs with different grid spacings.

The several runs did not cover all possible situations. For example, we made no attempt to determine whether the form of K(h) interacts with factors like geometry and boundary conditions to affect grid spacing. However, the data were sufficient for us to make several tentative conclusions regarding finite difference mesh design for unsaturated soil water flow modeling:

1 Spatial grid spacing is a function of curvature of equipotential lines and of the spatial rate of change of hydraulic gradient in the direction of flow.

2 Computational savings can be realized with very little loss of accuracy by designing an irregular mesh where the smallest grid sizes are used in regions of the expected sharpest curvature of equipotentials or greatest rate of change of gradient. Larger grid sizes may be used elsewhere.

3 After taking conclusions 1 and 2 into account, the actual dimensions of grid intervals is a matter of scale. For reasonably accurate finite difference modeling, a unit of length of a section may require subdivision into n grid intervals, regardless of whether the overall dimension of the unit is 1 or 10 m. This must not be construed to mean that the solutions to geometrically similar sections will be scaled reproductions of each other.

4 Much more numerical investigation is needed to come to firm conclusions regarding time and space grids for transient flow. Seemingly, both time intervals and grid intervals may be increased as a solution progresses.

5 Although the ADI method of finite differencing is unconditionally stable when applied to linear processes, it is unstable under certain conditions with nonlinear phenomena. The use of ADI techniques is not seriously hampered, however, because one must adjust Δt only for short periods either at the start of a solution or after an abrupt change in boundary conditions while a solution is progressing.

6 For the nonlinear equations which express unsaturated soil water flow, order of accuracy for square grids was nearer to being a function of grid interval size than to the square of grid interval size, as predicted by linear theory.

7 Desired accuracy is a function of individual preference and the parameters of given situations, but grids that are manifestly too coarse in either space or time are easily recognizable in the results. If Δx and Δy are too large, equipotentials exhibit irregular shapes. If Δt is too large, model outputs such as infiltration rates fluctuate with time. For Δt which is only moderately too large the fluctuations damp out, so that only initial estimates of the quantity will be affected.

Only two sections were considered in the study but they were quite different in shape and in boundary conditions. Based not only on the experience with them but also on less-documented experience with other cases, we feel that our conclusions in this investigation may be used as a general guide in designing or modifying finite difference grids. Continuing experience with finite difference applications may modify the above conclusions somewhat and may eventually lead to clearer understanding of discretization errors in nonlinear finite differencing with mixed boundary conditions.

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