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MATHEMATICAL MODELING OF EROSION[1] ON UPLAND AREAS

Mathematical Models
Erosion
Rainfall

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Abstract

Erosion due to rainfall on upland areas was mathematically described by the continuity-of-mass transport equation and relationships for interrill sediment detachment and the interaction between flow detachment and sediment load. The fluid flow is described by the kinematic wave equations.

The flow detachment and transport capacities are taken to be power functions of the flow depth. The interrill detachment is assumed to be proportional to the effective precipitation rate.

The resulting two dimensional equations described the time-space distribution of sediment concentration and transport rate for the rising limb and steady state portion of the runoff hydrograph.

Resume

On a décrit l'erosion à cause de pluie sur les hautes terres par la balance de mass et les equations pour le détachment de sédiment entre les ruisselets et l'interaction entre le détachment d'ecoulement et la quantité de sédiment. On a décrit l'ecoulement d'eau par les equations d'ongles cinématiques.

Le détachment d'ecoulement et les capacités de transport ont été décrit par les fonctions de puissance de la profondeur de ecoulement. On a fait d'assumer le détachment entre les ruisselets est par rapport au débit de pluie.

On obtient les equations en deux dimensions qui décrivent la distribution en temps et position de la concentratren de sédiment et du débit de transport pour la branche ascendante l'equilibre partie d'un hydrogramme d'ecoulement.

1 Contribution of the North Central Region, Agricultural Research service, U.S.D.A. in cooperation with the Missouri Agricultural Experiment Station and the Civil Engineering Department, University of Missouri, Columbia, Missouri.

BASIC EQUATIONS

The motion of the fluid and the sediment across the land are described by the equations for conservation of mass, momentum and energy. For this problem the material is treated as if it is traveling in a sheet across the land surface. This is obviously an incorrect hypothesis. The deviation of the hypothesis from reality is accounted for in various coefficients which are determined experimentally.

The equations will be derived for a strip of surface of unit width. A longitudinal view of the surface is shown in Fig. (1). Conservation of mass requires that for the fluid

$$[q - \frac{\partial q}{\partial x}\frac{\Delta x}{2}]\,\Delta t + \sigma\,\Delta x\,\Delta t - [q + \frac{\partial q}{\partial x}\frac{\Delta x}{2}\,\Delta t = \Delta h\,\Delta x \qquad (1$$

and for the sediment

$$[cq - \frac{\partial(cq)}{\partial x}\frac{\Delta x}{2}]\,\Delta t + E_R\,\Delta x\,\Delta t + E_I\,\Delta x\,\Delta t - [cq + \frac{\partial(cq)}{\partial x}\frac{\Delta x}{2}]\,\Delta t = \Delta(ch)\Delta x \qquad (2)$$

in which: q = flow rate per unit width, L^2/T; x = length coordinate, (L); t = time coordinate, (T); σ = effective precipitation rate, L/T; h = depth of flow, (L); c = concentration of sediment, F/L^3; E_R = rate of erosion per unit length due to flow, F/L^2T; E_I = rate of interrill erosion per foot, F/L^2T.

Algebraic manipulation of Eq. (1) and Eq. (2) results in

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = \sigma \qquad (3)$$

and

$$\frac{\partial(ch)}{\partial t} + \frac{\partial(cq)}{\partial x} = E_R + E_I \qquad (4)$$

The momentum equation is approximated by the relation

$$q = \alpha h^m \qquad (5)$$

as in kinematic wave analysis of the flow. α and m are constants.

The relationships describing the rates of rill and interrill erosion, E_R and E_I, must also be established.

Foster and Meyer[2] suggest that

$$\frac{E_R}{D_c} + \frac{cq}{T_c} = 1 \qquad (6)$$

in which: D_c = detachment capacity of the flow, $F/(L^2T)$; T_c = transport capacity of the flow, $F/(LT)$.

detachment capacity can be related to the shear stress on the bed, τ_o, as

$$D_c = a\,\tau_o^n = a(\gamma S_o)^n(h)^n \qquad (7)$$

in which a is a constant of proportionality, γ is the fluid specific weight, F/L^3, and S_o is the slope of the bed, and n is a constant.

Foster and Meyer[3] further suggest that the transport and detachment capacities are related by a constant such that

2 Foster, G. R. and L. D. Meyer, "A Closed Form Erosion Equation for Upland Areas," Sedimentation: A Symposium to Honor Professor H. A. Einstein. H. W. Shen, Ed., Fort Collins, Colorado, U.S.A., 1971.

3 Ibid.

$$T_c = \frac{1}{K_R} D_c = \frac{1}{K_R} a(\rho g S_o)^n h^n = \beta h^n$$

Then, with Eqs. (6, 7 and 8)

$$E_R = K_R(\beta h^n - cq)$$

The interrill contribution, E_I, has been discussed in detail by Foster, Meyer and Onstad[4]. They indicate that E_I varies with the rainfall intensity, soil and cover conditions and with slope steepness. The relationship can be expressed as

$$E_I = K_I \sigma \tag{10}$$

Combining Eqs. (3), (4), (5), (9) and (10) results in

$$h \frac{\partial c}{\partial t} + q \frac{\partial c}{\partial x} = K_R(\beta h^n - cq) + K_I \sigma - \sigma c \tag{11}$$

OVERLAND FLOW

The solutions for the flow equations for constant rainfall excess have been given by several investigators, see for example Eagleson[5]. The results, for constant effective rainfall rate are

$$h = \sigma t$$

$$q = \alpha h^m = \alpha \sigma^m t^m \tag{12}$$

along a characteristic defined by

$$x - x_o = \alpha \sigma^{m-1} (t - t_o)^m \tag{13}$$

Several results are important in the analysis of the sediment transport. The time of concentration for the flow is given by

$$t_c = (\frac{L}{a})^{1/m} \sigma(1/m)-1 \tag{14}$$

The flow solution can be divided into two regions in the x-t plane: the zone of flow establishment and the zone of established flow, as shown in Fig. (2). In the zone of flow establishment the discharge is a function of time alone, whereas in the zone of established flow the discharge is a function of space alone.

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4 Foster, G. R., L. D. Meyer, and C. A. Onstad, "Erosion Equations Derived From Modeling Principles," American Society of Agricultural Engineers Proc., Paper 73-2550, presented at National meeting, Dec. 1973, Chicago.

5 Eagleson, P. S., Dynamic Hydrology, McGraw-Hill Book Company, New York, 1970.

SEDIMENT TRANSPORT IN ZONE OF FLOW ESTABLISHMENT

The sediment transport equations (11) can be solved in two parts corresponding to the flow zones. In the zone of flow establishment the equation to be solved is

$$\frac{\partial c}{\partial t} + \alpha\sigma^{m-1}t^{m-1}\frac{\partial c}{\partial x} = K_R(\beta\sigma^{n-1}t^{n-1} - c\alpha\sigma^{m-1}t^{m-1}) + \frac{K_I}{t} - \frac{c}{t} \qquad (15)$$

in which the flow relationship $\sigma = \alpha(\sigma t)^m$ has been used. This can be expressed as the ordinary differential equation

$$\frac{dc}{d\eta} + [K_R\alpha(\sigma\eta)^{m-1} + \frac{1}{\eta}]c = K_R\beta(\sigma\eta)^{n-1} + \frac{K_I}{\eta} \qquad (16)$$

in which the coordinate η defines the characteristic lines for the solution and is given by the curves

$$\eta - \eta_o = t - t_o$$
$$x - x_o = \frac{\alpha\sigma^{m-1}}{m}(t^m - t_o^m) \qquad (17)$$

Some of these characteristics are displayed in Fig. (2) as dashed lines.

Equation (16) can be made a perfect differential using the integrating factor

$$p = \eta\,\exp[K_R\alpha\sigma^{m-1}\eta^m/m] \qquad (18)$$

and the integral for equal flow and transport capacity exponents, m=n, and η_o=0, becomes

$$pc = K_R\beta\sigma^{m-1}\int \eta^m \exp[\frac{K_R L}{m}(\frac{\eta}{t_c})^m]d\eta + K_I\int \exp[\frac{K_R L}{m}(\frac{\eta}{t_c})^m]d\eta \qquad (19)$$
$$+ \text{constant}$$

in which t_c is the time of concentration for the flow, Eq. (14). The first integral can be integrated by parts to yield

$$c = \frac{\beta}{\alpha} - [\frac{\beta}{\alpha} - K_I]\,\frac{1}{\zeta}\,e^{-\zeta^m}\int_0^{\zeta} e^{u^m}du + (\text{constant})\,\frac{1}{\zeta}\,e^{-\zeta^m} \qquad (20)$$

with the variable ζ given by

$$\zeta^m = \frac{K_R L}{m}(\frac{\eta}{t_c})^m \qquad (21)$$

The constant of integration must be zero in order for the concentration to be finite at η = 0, so the solution is

$$c = \frac{\beta}{\alpha} - [\frac{\beta}{\alpha} - K_I]\,\frac{1}{\zeta}\,e^{-\zeta^m}\int_0^{\zeta} e^{u^m}du \qquad (22)$$

The integral can be expressed as a infinite series. The series terminates after a few terms for those special cases in which m = 1/N in which N is an integer. In most cases, however, m is greater than one. For the particular value of m=2 the integral is tabulated as Dawson's integral.[6]

The sediment transport rate, g, is given by

$$g = cq \qquad (23)$$

[6] Abramowitz, M. and I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Applied Math Series, 55, Washington, D.C., 1964.

which can be non-dimensionalized by dividing by the transport capacity at the outfall for equilibrium conditions. From Eq. (8)

$$(T_c)_e = \beta(\sigma t_c)^m = \beta\sigma L/\alpha \tag{24}$$

Thus, the sediment transport rate can be expressed as

$$\frac{g}{(T_c)_e} = c\,\frac{\alpha}{\beta}\left(\frac{t}{t_c}\right)^m = \left(\frac{t}{t_c}\right)^m - \left[1 - \frac{K_I}{(\beta/\alpha)}\right]\left(\frac{t}{t_c}\right)^{m-1} \cdot \left(\frac{m}{K_R L}\right)^{1/m} e^{-\zeta^m} \int^{\zeta} e^{u^2} du \tag{25}$$

Equation (25) is valid in the zone of flow establishment for the case where the flow and erosion exponents, m and n, are equal. For exponents greater than unity, the path of the solution crosses the boundary between the zone of flow establishment and the zone of established flow. The solution must be continued considering the change in flow conditions.

SEDIMENT TRANSPORT IN THE ZONE OF ESTABLISHED FLOW

The partial equation for the sediment concentration Eq. (11) in the zone of established flow can be expressed

$$\frac{(\sigma x)^{\frac{1}{m}-1}}{\alpha^{1/m}}\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = K_R\beta\alpha^{-n/m}(\sigma x)^{\frac{n}{m}-1} - K_R c + \frac{K_I}{x} - \frac{c}{x} \tag{26}$$

in which the flow relationship $\sigma = \sigma x$ has been used. This can be expressed as the ordinary differential equation

$$\frac{dc}{d\xi} + \left(K_R + \frac{1}{\xi}\right)c = K_R\beta\alpha^{-n/m}(\sigma x)^{\frac{n}{m}-1} + K_I\,\frac{1}{x} \tag{27}$$

in which the coordinate, ξ, defines the characteristics for the solution and is given by the curves

$$\xi - \xi_o = x - x_o$$

$$t - t_o = m\,\frac{\sigma^{\frac{1}{m}-1}}{\alpha^{1/m}}\left(x^{1/m} - x_o^{1/m}\right) \tag{28}$$

Some of these paths are also displayed in Fig. (2) as dashed lines.

The ordinary differential equation can be converted to a perfect differennential using the integrating factor

$$p^* = \xi\, e^{K_R\xi} \tag{29}$$

The solution can be expressed, for m=n, as

$$c = \frac{\beta}{\alpha} - \left(\frac{\beta}{\alpha} - K_I\right)\frac{1}{K_R L(x/L)} + (\text{constant})\,\frac{1}{x/L}\,e^{-K_R L(x/L)} \tag{30}$$

The constant of integration is determined using the condition that the solutions must match at the boundary between the zone of flow establishment and the zone of established flow. Let x_b, t_b = coordinates of boundary, and c_b = concentration on boundary. The resulting solution is

$$c = \frac{\beta}{\alpha} + \left(c_b - \frac{\beta}{\alpha}\right)\frac{x_b/L}{x/L}\exp[-K_R L(x-x_b)/L] - \left(\frac{\beta}{\alpha} - K_I\right)\frac{1-\exp[-K_R L(x-x_b)/L]}{K_R L} \tag{31}$$

The corresponding sediment transport rate is given by

$$\frac{g}{(T_c)_e} = \frac{x}{L} - \left(1 - \frac{c_b}{\beta/\alpha}\right)\frac{x_b}{L}\exp[-K_R L(x-x_b)/L] - \left(1 - \frac{K_I}{\beta/\alpha}\right)\left(\frac{x}{L}\right)\frac{1-\exp[-K_R L(x-x_b)/L]}{K_R L} \tag{32}$$

SEDIMENT TRANSPORT AT EQUILIBRIUM

Sediment transport achieves steady state when both of the following conditions are met:

(1) The solution path for the flow starting from $x = 0$ at $t = 0$ reaches the outfall. This defines the time of concentration

$$t_c = \left(\frac{L}{\alpha}\right)^{1/m}\sigma^{\frac{1}{m}-1} \tag{33}$$

(2) The solution path for the sediment starting from $x = 0$ at $t = 0$ reaches the outfall. This time is given by

$$t_s = \frac{L}{\ }\Big)^{1/m}\sigma^{\frac{1}{m}-1} \tag{34}$$

Note that the time for condition (2) is m times the time of concentration.

EXAMPLE AND SUMMARY

At equilibrium, Eq. (33) reduces to

$$\frac{g}{(T_c)_e} = \frac{x}{L} - \left(1 - \frac{K_I}{\beta/\alpha}\right)\frac{1-\exp[-K_R L(x/L)]}{K_R L} \tag{35}$$

Foster and Meyer[7] established this equilibrium relation and compared it with the experimental results of Young and Mutchler.[8] Excellent agreement was found using

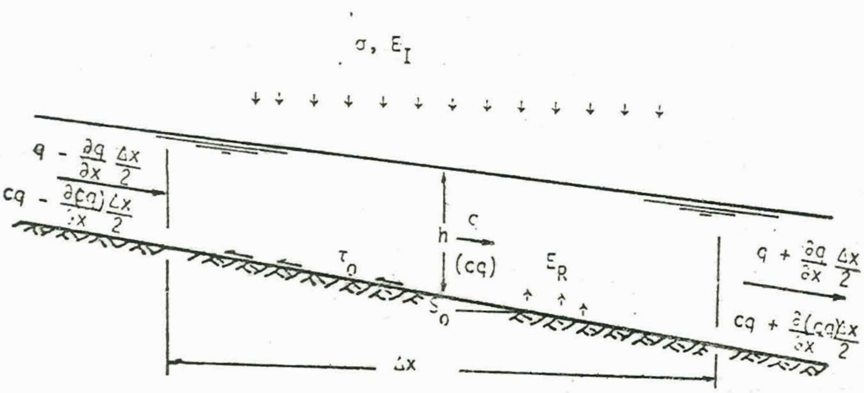
$$K_I(\beta/\alpha) = 0.15\ ;\quad K_R L = 0.28 \tag{36}$$

These same parameters were used to evaluate the non-equilibrium sediment transport equations, and, for m=2, the results are displayed in Fig. (3).

The basic equations can be used as the basis of a model of soil movement across a watershed. Application to an actual watershed will require numerical solution of the differential equations. The results of this analytical solution provide a check of the numerical procedure and should help in locating the source of computational difficulties.

Op. cit.

Young, R.A., and C.K. Mutchler, "Soil Movement on Irregular Slopes." Water Resources Research, 5(5), 1969.



1: Overland flow model and symbols for unit width.

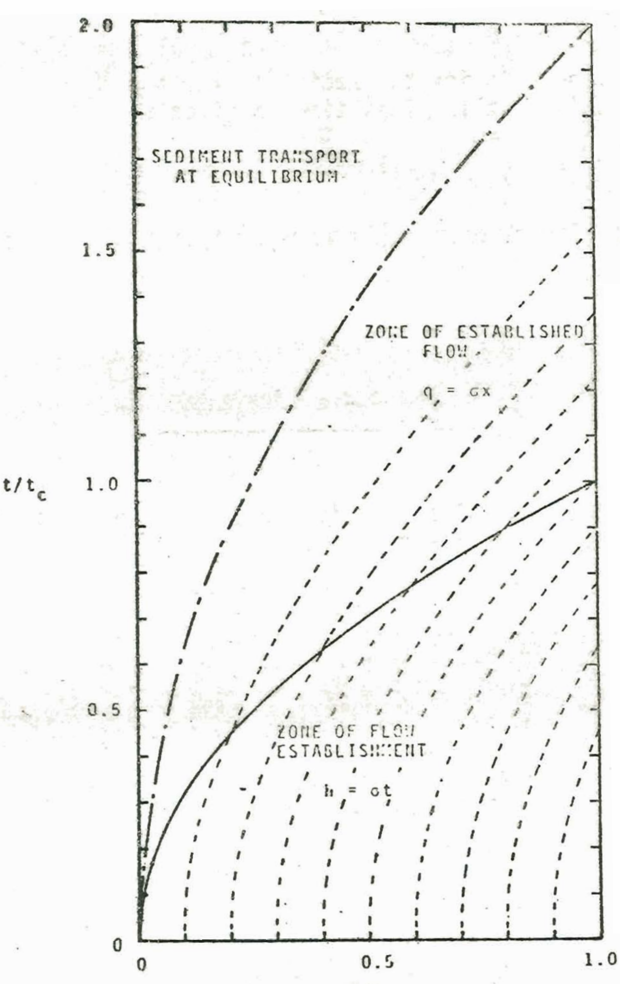
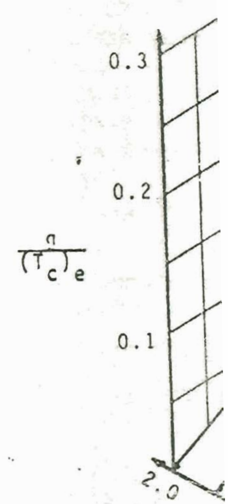


Fig. 2: Characteristic lines for solution of sediment motion equations. m = 2



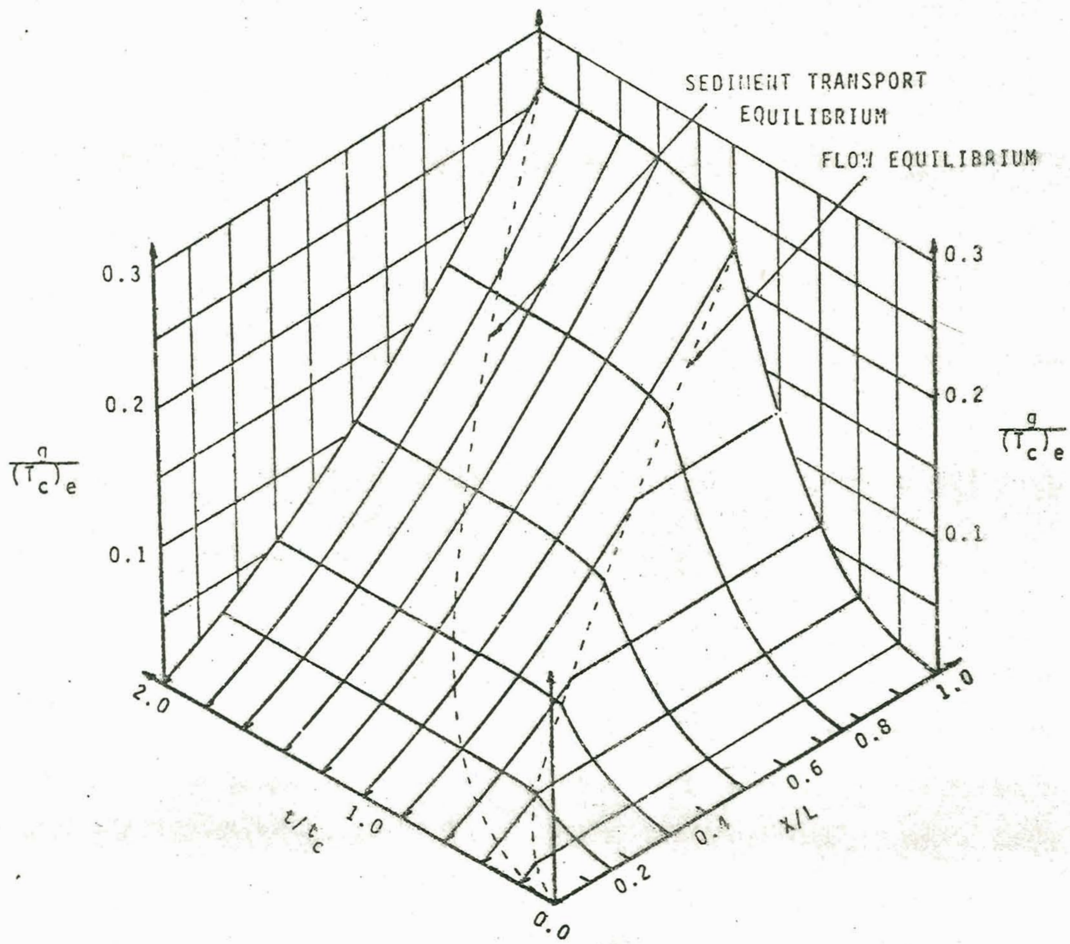


Fig. 3: Sediment transport rate as a function of time and space for $m = 2$, $K_T/(8/\alpha) = 0.15$, and $K_D L = 0.28$.