Agricultural Meteorology, 15(1975) 343—353

SENSITIVITY ANALYSES OF THE COMBINATION EVAPOTRANSPIRATION EQUATION

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(Received March 31, 1975; accepted August 25, 1975)

ABSTRACT

Saxton, K. E., 1975. Sensitivity analyses of the combination evapotranspiration equation. Agric. Meteorol., 15: 343—353

Sensitivity equations were derived by differentiating the combination energy budget—aerodynamic evapotranspiration equation with respect to each variable. The sensitivity coefficients of these equations define the change in computed PET due to a change in the variable. Daily sensitivity coefficients were determined by applying two years of data, March through November, obtained in western Iowa over corn and grass. Annual graphs of the sensitivity coefficients showed their mean values, daily variation, and annual trends.

The results showed computed evapotranspiration most sensitive to net radiation. During midyear, PET values usually change 50—90% of any radiation change or error, whereas only 20—30% of any change of vapor pressure deficit or wind travel transfers to the PET value. In the spring and fall months, the aerodynamic portion of the equation played a larger role; thus, the net radiation coefficients decreased and the aerodynamic variable coefficients increased. The calculated PET values were not largely sensitive to the wind profile parameters (Z_a, d, and Z_o) but experience has shown that large errors can occur in the measurement or prediction of these values; thus, their effect on PET can still be significant in many cases.

INTRODUCTION

To understand the characteristics of a mathematical equation or model, we must first understand each variable, then know its relative role in the model. Each variable may be studied individually to understand such items as magnitudes, variability, diurnal variations, and measurement accuracy; however, to understand the relative role of each variable requires a sensitivity analysis.

By definition, a sensitivity analysis shows the effect of change of one factor on another (McCuen, 1973). If the change of the dependent variable of an equation or model is studied with respect to change in each of several independent variables, the sensitivity coefficients will show the relative importance of each of the variables to the model solution. The application of sensitivity analyses to several hydrologic situations was recently demonstrated by McCuen (1973).

This study considered the sensitivity of the variables in the combination energy budget—aerodynamic equation for estimating potential evapotranspiration (PET). The combination equation is presented in its working form, with a discussion of applied data. Then appropriate sensitivity equations are derived and example solutions presented. The resulting sensitivity coefficients show the sensitivity of the daily PET to each of the seven variables throughout the March-to-November study period.

THE COMBINATION EQUATION

The combination energy budget—aerodynamic equation for estimating potential ET is based on the derivation by Penman (1948) but includes recent modifications to allow for measured net radiation and the incorporation of the turbulent wind profile diffusion theory (Businger, 1959; Tanner and Pelton, 1960; Van Bavel, 1966). A complete derivation of the equation was presented by Saxton (1972).

The combination equation, as expressed for this study, is:

$$E = \frac{(\Delta/\gamma)\, R_n + 7.12\, d_a\, u_a\, W^{-1}}{(1 + \Delta/\gamma)\, 583} \tag{1}$$

and:

$$W = \left[\ln\left(\frac{Z_a - d}{Z_0}\right)\right]^2 \tag{2}$$

where E = potential evapotranspiration (cm day^{-1}); Δ = slope of psychrometric saturation line (mb °C^{-1}); γ = psychrometric constant (mb °C^{-1}); R_n = net radiation (cal cm^{-2} day^{-1}); d_a = vapor pressure deficit (mb); u_a = horizontal wind movement at elevation Z_a (km day^{-1}); W = wind profile coefficient; Z_a = anemometer height above soil (cm); d = wind profile displacement height (cm); Z_0 = wind profile roughness height (cm).

The numerical constants in the equation represent unit conversions plus selected standard values for air density (1.168 · g cm^{-3}), heat of vaporization (583 cal. g^{-1}), Von Karman coefficient (0.41), barometric pressure (1,000 mb), water/air molecular ratio (0.622), and the psychrometric constant (0.66).

INSTRUMENTATION AND DATA

Instruments were operated on corn and bromegrass research watersheds near Treynor, Iowa (Saxton et al., 1971) to obtain daily values of the variables in eq. 1. The sensors were maintained approximately 1 m above the soil, or 1 m above the top of the crop canopy when a canopy was present, and were located about 60 m (200 ft.) on either side of a common watershed boundary separating corn and bromegrass watersheds. The sensors over corn were mounted on a vertically movable cradle attached to a stationary tower; those over grass were permanently positioned.

Net radiation was measured with miniature net radiometers (Fritschen, 1960, 1965) which were recorded individually on strip-chart recorders with mechanical integrators. Air temperature was sensed by ventilated thermocouples, air humidity by ventilated lithium-chloride dew cells, and wind travel by 3-cup aluminium anemometers. These sensors over both crops were continuously recorded by a multipoint strip-chart recorder. Measurements were made from about March 15 to December 1 during 1969 and 1970.

To obtain daily solutions of eq. 1, daily net radiation was summarized from sunup to sunup. This included measurements of the night-time outgoing radiation which largely compensated for daily soil heat storage, which was not measured. Wind travel only during daylight hours was applied because air movement contributes to potential ET in proportion to vapor pressure deficit, which becomes nearly zero during most nights at the Treynor, Iowa, location. The required correspondence of wind travel and vapor pressure deficit was shown by Tanner and Pelton (1960). An average daylight (06h00—18h00) vapor pressure deficit was obtained by first calculating vapor pressure deficits from air temperature and dew probe readings at 06h00, 10h00, 14h00, and 18h00, then weighting these vapor pressure deficits by 1, 2, 2, and 1, respectively. A study of 24 days showed that these weighted averages very closely approximated daylight averages computed from readings at 1-h intervals.

It was not feasible to measure wind profiles for obtaining representative d and Z_0 values. Several locations would have been required to sample the spatial variation over the watersheds, and the data and computations would have been voluminous. In addition, the wind profile theory contains assumptions that are questionable when applied to undulating surfaces with tall crops, open canopies, and thermal instabilities. After several unsuccessful attempts to estimate d and Z_0 values from reported relationships (Lemon, 1963; Szeicz et al., 1969), an annual distribution of values for the wind profile coefficient W (eq. 2) was estimated such that they provided realistic PET values for one year's data. These estimated W values were then applied to two subsequent years' data with equal success. Values for the wind parameters d and Z_0 were obtained for the sensitivity analyses by using the estimated W values, the measured instrument height Z_a, a relationship for d based on canopy height and density, and eq. 2 (Saxton, 1972; Saxton et al., 1974).

DEVELOPMENT OF SENSITIVITY EQUATIONS

Sensitivity analyses may be conducted by two separate, but related, techniques. One method is to obtain a solution from a set of variable values, then increase each variable value a small amount while holding all other values constant and note the change of the solution. A better approach is to mathematically differentiate the equation or model under study to derive equations for the rate of change of the independent variable with respect to each dependent variable. This is the more efficient method, but it requires that the model be mathematically tractable. The differentiation method was applied to the combination equation as follows.

Scarborough (1958) shows that sensitivity equations (also called error equations) can be developed for a function:

$$N = f(u_1, u_2, \ldots, u_n) \tag{3}$$

by first writing:

$$N + \Delta N = f(u_1 + \Delta u_1, u_2 + \Delta u_2, \ldots) \tag{4}$$

Then, applying Taylor's theorem and neglecting squares, products, and higher powers in the expansion yields:

$$\Delta N = \frac{\partial N}{\partial u_1} \Delta u_1 + \frac{\partial N}{\partial u_2} \Delta u_2 + \ldots \tag{5}$$

Relative changes or errors can be defined as:

$$N_\xi = \frac{\Delta N}{N} \tag{6}$$

$$u_\xi = \frac{\Delta u}{u} \tag{7}$$

Substituting eqs. 6 and 7 into 5 provides the general equation:

$$N_\xi = \left(\frac{\partial N}{\partial u_1} \frac{u_1}{N}\right) u_{1\xi} + \left(\frac{\partial N}{\partial u_2} \frac{u_2}{N}\right) u_{2\xi} + \ldots \tag{8}$$

which expresses the relative change of N with respect to the sum of the relative changes of each variable. If we consider change or error that occurs in only one variable, all other terms would go to zero, leaving, for example:

$$N_\xi = \left(\frac{\partial N}{\partial u_1} \frac{u_1}{N}\right) u_{1\xi} \tag{9}$$

The bracketed terms become a dimensionless coefficient which expresses the percentage of the relative variable change transmitted to the relative dependent variable. This may be defined as the sensitivity coefficient, For example, a sensitivity coefficient of 0.2 would show that a 10% change in u_1 ($u_{1\xi}$ = 0.10) would cause a 2% change in N (N_ξ = 0.02).

General eq. 9 was applied to the combination model (eqs. 1 and 2 combined) for each of the seven variables. The resulting equations are presented in Table I; the bracketed terms form the sensitivity coefficients. The same expression holds for the three aerodynamic variables (d_a, u_a, and W) because of their similar position in the model.

Equations were developed for the effect of the wind profile parameters (Z_a, d, and Z_0) on the E values (eqs. 1 and 2) and on just the wind coefficient W (eq. 2); however, results are not shown for the effect on W because wind profiles were not measured in this study. The equation representing all variables is the combined effect as suggested by eq. 8. If all variables had the same relative error, the ξ terms could be factored and the sensitivity coefficient would become the sum of the individual coefficients.

SOLUTION OF THE SENSITIVITY EQUATION

Each of the sensitivity coefficients of Table I contains the values for all other variables; thus, the sensitivity of any one value is quite dependent upon the values for all other variables. Little insight is gained if the coefficients are determined for mean variable values because only rarely would this combination occur; and if each variable were incremented over its expected range, the number of permutations would be great. Most of the variables exhibit seasonal trends; thus, the sensitivities would also be expected to vary seasonally. Therefore, the sensitivity equations were solved with observed daily variable values for 1969 and 1970 March-through-November study periods.

Two sets of data, one over corn and the other over grass, were applied for each of the two study periods. Example data over grass during the 1969 study period are shown in Figs. 1—4 for net radiation (R_n), delta/gamma (Δ/γ), vapor pressure deficit (d_a), and wind travel (u_a), respectively. The net radiation values show considerable daily variation and the expected solar curve of maxima. For the Δ/γ values, Δ is a function of mean ambient air temperature, and the Δ/γ distribution follows the seasonal temperature trends. The vapor pressure deficit values show a distinct annual trend, but they have large day-to-day variations. Wind travel is also highly variable and has a slight trend of higher values in the spring and fall. These measured data provided numerous value combinations due to the daily variation and seasonal trends typical of the western Iowa climate.

Daily solutions of the sensitivity equations are shown in Figs. 5 and 6 for the 1969 data over grass. Results were very similar for the 1969 data over corn and the 1970 data over grass and corn. Each sensitivity coefficient is a solution of the complete, bracketed term in the sensitivity equations (Table I) using the variable values for that day.

TABLE I

Sensitivity equations for the combination evapotranspiration equation

Variable	Sensitivity equation*
	Evapotranspiration (eqs. 1 and 2):
R_n	$E_\xi = \left\{ \dfrac{1}{\left[\dfrac{7.12\, d_a\, u_a\, W^{-1}}{(\Delta/\gamma)\, R_n}\right] + 1} \right\} R_{n\xi}$
Δ	$E_\xi = \left\{ \dfrac{(R_n - 7.12\, d_a\, u_a\, W^{-1})}{[\Delta\, R_n + \gamma\,(7.12\, d_a\, u_a\, W^{-1})]} \quad \dfrac{\gamma\, \Delta}{\gamma + \Delta} \right\} \Delta_\xi$
d_a, u_a, W	$E_\xi = \left\{ \dfrac{1}{\left[\dfrac{(\Delta/\gamma)\, R_n}{7.12\, d_a\, u_a\, W^{-1}}\right] + 1} \right\} d_{a\xi}$
Z_a	$E_\xi = \left\{ \dfrac{2\, W^{1/2}}{\left[\dfrac{(\Delta/\gamma)\, R_n}{7.12\, d_a\, u_a\, W^{-1}}\right] + 1} \quad \dfrac{1}{[1 - (d/Z_a)]} \right\} Z_{a\xi}$
	$E_\xi = \left\{ \dfrac{2\, W^{-1/2}}{\left[\dfrac{(\Delta/\gamma)\, R_n}{7.12\, d_a\, u_a\, W^{-1}}\right] + 1} \quad \dfrac{d}{Z_a - d} \right\} d_\xi$
Z_0	$E_\xi = \left\{ \dfrac{2\, W^{-1/2}}{\left[\dfrac{(\Delta/\gamma)\, R_n}{7.12\, d_a\, u_a\, W^{-1}}\right] + 1} \right\} Z_{0\xi}$
All	$E_\xi = K_1\,(R_{n\xi}) + K_2\,(\Delta_\xi) + K_3\,(d_{a\xi}) + K_4\,(u_{a\xi}) + K_5\,(Z_{a\xi}) + K_6\,(d_\xi) + K_7\,(Z_{0\xi})$
	where K_1 to K_7 are the sensitivity coefficients defined above for each variable
	Wind coefficient (eq. 2):
Z_a	$W_\xi = \left[2 \left(\dfrac{Z_a}{Z_a - d}\right) W^{-1/2} \right] Z_{a\xi}$
	$W_\xi = \left[2 \left(\dfrac{d}{Z_a - d}\right) W^{-1/2} \right] d_\xi$
Z_0	$W_\xi = \left(2\, W^{-1/2}\right) Z_{0\xi}$

* All symbols as defined for eqs. 1 and 2. ξ denotes relative errors as defined by eqs. 6 and 7.

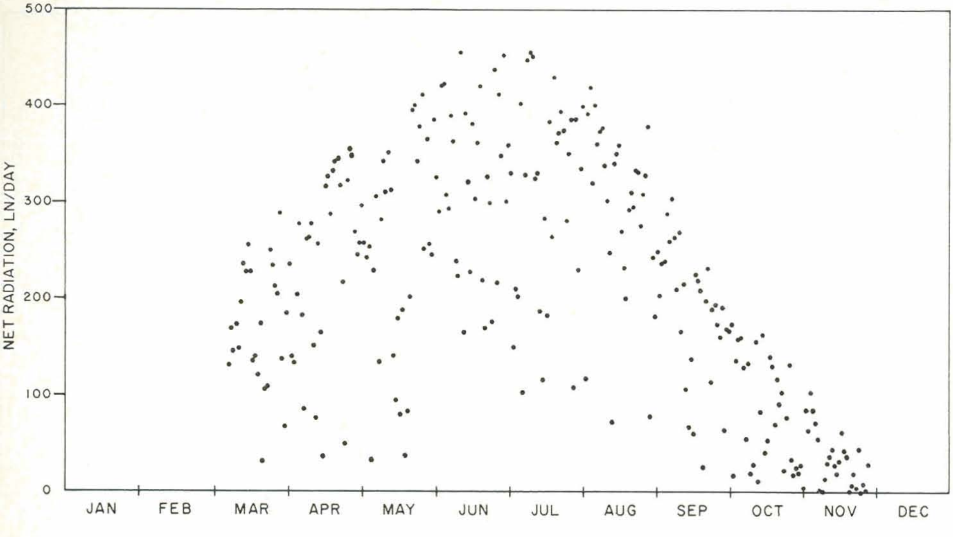

Fig.1. Daily total net radiation measured over grass, 1969.

As expected from the data, the coefficients for each variable have considerable day-to-day variation, and some display a seasonal trend. The Δ sensitivity coefficients average about 0.2, with a slight trend toward larger values early and late in the season. The potential evapotranspiration values are most sensitive to net radiation. Approximately 80% of any R_n change will be reflected in the E values during midyear. Each of the wind variables (d_a, u_a, and W) plays the next most significant role.

The sensitivity coefficients of net radiation and the aerodynamic variables have opposing seasonal trends that reflect their changing relative importance

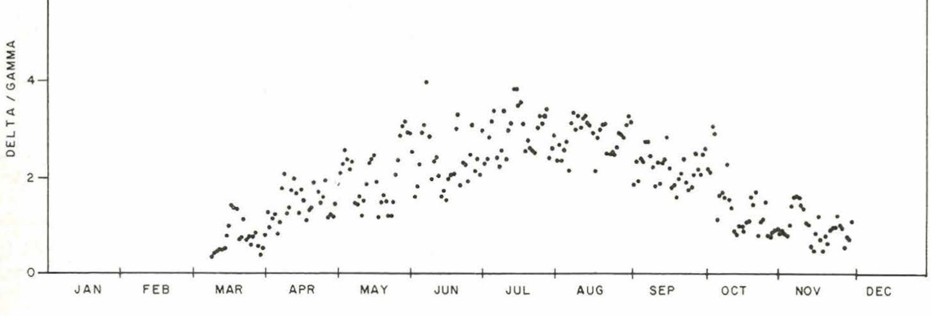

Fig.2. Mean daylight Δ/γ values measured over grass, 1969.

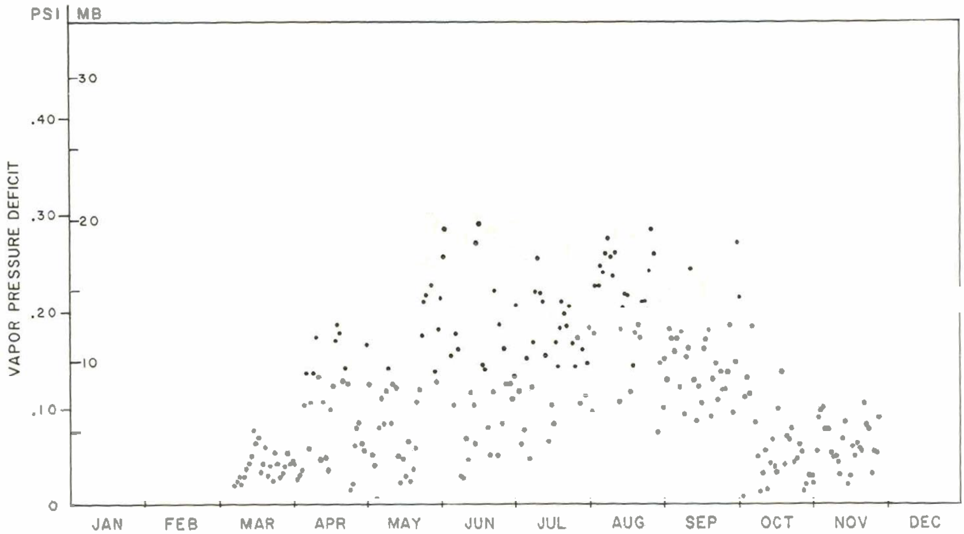

Fig. 3. Mean daylight vapor pressure deficit measured 0.4—0.8 m above grass, 1969.

to the computed E values. Net radiation is most significant during midseason, but the aerodynamic terms become more important early and late in the year. Others have noted the relative radiation—aerodynamic contribution, but

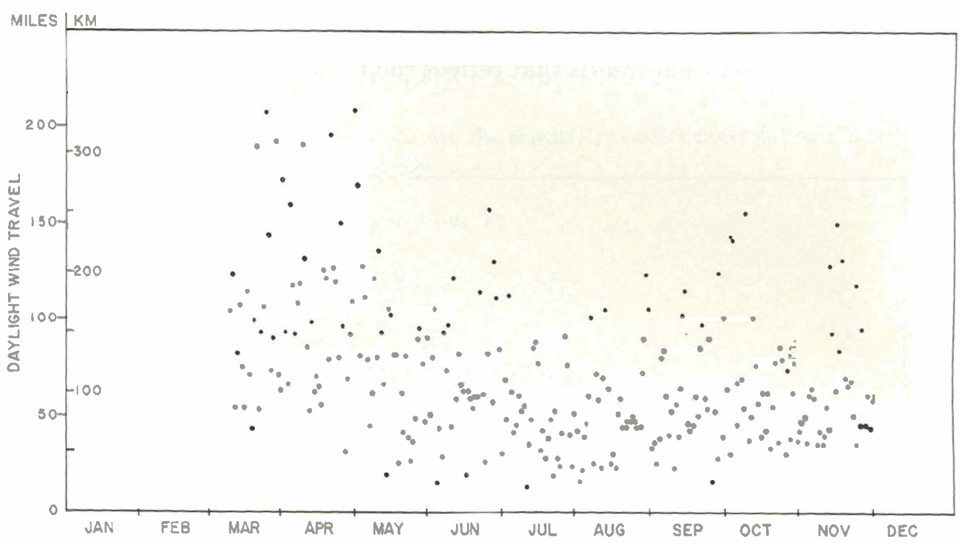

Fig. 4. Daylight wind travel 0.4—0.8 m above grass, 1969.

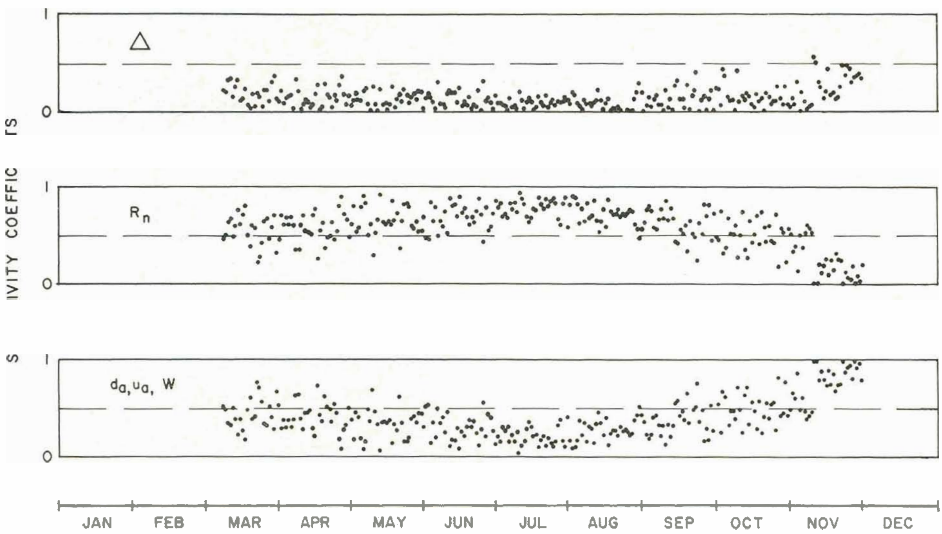

Fig. 5. Sensitivity coefficients for PET data over grass, 1969.

their data were limited to a few selected days and general discussions (Tanner and Pelton, 1960; Skidmore et al., 1969).

The sensitivity of E to the wind profile parameters (Z_a, d, and Z_0) is shown in the upper part of Fig. 6. Each variable has a moderately low sensitivity and a minor annual trend. Z_a has the largest coefficient, but it can be measured the most accurately. The Z_0 sensitivity of 0.1—0.2 agrees with that noted by Van Bavel (1966). Although the Z_0 sensitivity is small, this is a difficult variable to measure or estimate accurately, and errors of 200—300% can occur (Saxton, 1972; Munro and Oke, 1973), which would cause errors in the calculated E values of 20—60% during midyear and even greater errors early and late in the year.

The total sensitivity coefficients (bottom of Fig. 6) indicate the expected change in E if all variables should change a uniform amount in the same direction; this is improbable, but the values show potential magnitudes. Mid-season values averaged about 1.5; thus, a 5% error in all variables would result in a 7.5% error in the calculated E value. The total coefficients increased early and late in the season when the aerodynamic terms were more important.

These sensitivity solutions demonstrate the typical values, daily variation, and time trends. The mean sensitivity values can be combined with variable errors or measurement accuracy to estimate evaporation accuracy; or, conversely, required measurement accuracy can be estimated for a desired result accuracy. With several variables involved, a unique solution is impossible,

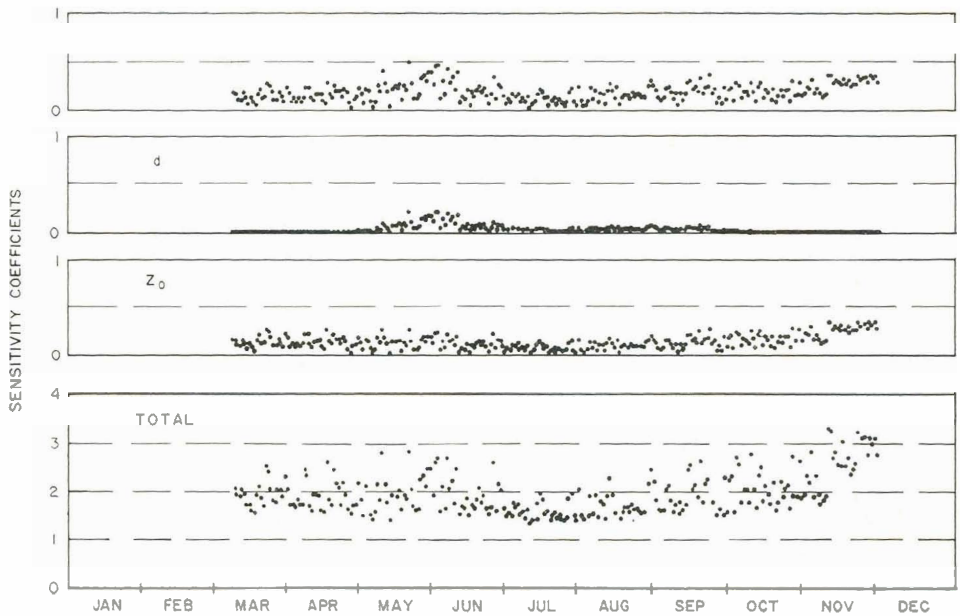

Fig.6. Sensitivity coefficients for PET data over grass, 1969.

but combining the sensitivities and accuracies shows which variables need to be measured most accurately and the probable result of their combined error.

ACKOWLEDGEMENT

A contribution of the Iowa-Missouri Area, North Central Region, Agricultural Research Service, United States Department of Agriculture, in cooperation with the Iowa Agriculture and Home Economics Experiment Station, Iowa State University, Ames, Iowa.

REFERENCES

Businger, J. A., 1959. A generalization of the mixing length concept. J. Meteorol., 16: 516—523.

Fritschen, L. J., 1960. Construction and calibration details of the thermal transducer-type net radiometer. Bull. Am. Meteorol. Soc., 41: 180—183.

Fritschen, L. J., 1965. Miniature net radiometer improvements. J. App. Meteorol., 4: 528—532.

Lemon, E. R., 1963. The energy budget at the earth's surface, II. U.S. Dept. Agric., Agric. Res. Serv., Prod. Res. Rep., 72.

McCuen, R. H., 1973. The role of sensitivity analysis in hydrologic modeling. J. Hydrol., 18: 37—53.
Munro, D. S. and Oke, T. R., 1973. Estimating wind profile parameters for tall dense crops. Agric. Meteorol., 11: 223—228.
Penman, H. L., 1948. Natural evaporation from open water, bare soil, and grass. Proc. R. Soc. Lond., Ser. A, 193: 120—146.
Saxton, K. E., 1972. Watershed Evapotranspiration by the Combination Method. Unpublished Ph.D. Thesis, Iowa State Univ. Lib., Ames, Iowa. (Dissertation Abstracts Inter. 33(4): 1514b—1515b Oct. 1972. Order No. 72—26, 940.)
Saxton, K. E., Johnson, H. P. and Shaw, R. H., 1974. Watershed evapotranspiration estimated by the combination method. Trans. Am. Soc. Agric. Eng., 17(4): 668—672.
Saxton, K. E., Spomer, R. G. and Kramer, L. A., 1971. Hydrology and erosion of loessial watersheds. Am. Soc. Civ. Eng. Proc. J. Hydrol. Div., 97(HY11): 1835—1851.
Scarborough, J. B., 1958. Numerical Mathematical Analysis. The Johns Hopkins Press, Baltimore, Md.
Skidmore, E. L., Jacobs, H. S. and Powers, W. L., 1969. Potential evapotranspiration as influenced by the wind. Agron. J., 61: 543—546.
Szeicz, G., Endrödi, G. and Tajchman, S., 1969. Aerodynamic and surface factors in evaporation. Water Resour. Res., 5(2): 380—394.
Tanner, C. B. and Pelton, W. L., 1960. Potential evapotranspiration estimates by the approximate energy balance method of Penman. J. Geophys. Res., 65(10): 3391—3413.
Van Bavel, C. H. M., 1966. Potential evapotranspiration: the combination concept and its verification. Water Resour. Res., 2(3): 455—467.