General Stochastic Unit Hydrograph

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Abstract: Unit hydrographs for a gaged site can be determined from observed events. If several events are used for the derivation, several unit hydrographs result. An averaging procedure must be used to achieve a single representative unit hydrograph. The variability is due to inherent uncertainty in the rainfall processes and to inadequacies in the basic model. It is appropriate to consider the unit hydrograph itself a random function. This paper develops a stochastic expression for the instantaneous unit hydrograph (IUH) based upon the Nash cascade. This conceptual model holds the number of reservoirs constant, while treating the reservoir constant as a random variable. Records of 24 storm events, observed on a 12.2-km^2 watershed located in north central Missouri, were studied. The results of the study indicate that the stochastic model can be used to estimate the hydrograph.

INTRODUCTION

Unit hydrographs are important tools in design hydrology. Often one is faced with the problem of producing a unit hydrograph for a gaged watershed. Producing such a unit hydrograph for a gaged watershed also results in the problem of an embarrassment of riches. Unit hydrographs derived from several rainfall-runoff events result in several different unit hydrographs. These differences result from errors in the observed rainfall and runoff from which the unit hydrograph was derived and from inadequacies in the linear system theory on which the unit hydrograph is based. Regardless of the cause, a representative "average" unit hydrograph must be selected. This selection is usually achieved by eye. In this paper, a mathematical approach based upon stochastic unit hydrographs will be described.

Conceptual models have long been used to describe watershed hydrologic response. Modeling watershed response using a linear system approach was introduced by Zoch (1934, 1936, 1937). He used a linear reservoir to describe the runoff process. This linear reservoir approach was expanded by Nash (1957) through the use of a cascade of identical linear reservoirs to represent a watershed. The result was a Gamma function representation of the unit hydrograph

$$q(t) = \frac{1}{k(n-1)!}\left(\frac{t}{k}\right)^{(n-1)} e^{(-t/k)} \quad \text{(1)}$$

in which q = discharge, t = time, n = number of reservoirs, and k = reservoir constant. In general, when fitting a gamma unit hydrograph to several rainfall-runoff events on a watershed, one finds a wide variability in n, the number of reservoirs, and in k, the reservoir constant. These parameters should be treated as random quantities. Sarino and Serrano

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(1990) presented a stochastic model which treated the parameter k as a random variable and derived the stochastic unit hydrograph when the number of reservoirs in the cascade, n, is less than or equal to two.

This paper extends the methodology presented by Sarino and Serrano (1990). A general form for the stochastic instantaneous unit hydrograph is derived. The theory is applied to a small watershed. Twenty-four storm events are investigated with the parameters n and k for each storm event determined using an optimization technique. The statistical features of the parameter k are displayed, and a comparison of the stochastic model with the classic approach for fitting the unit hydrograph to the study watershed is given.

STOCHASTIC INSTANTANEOUS UNIT HYDROGRAPH

The watershed runoff process is assumed to be a linear dynamic system that can be represented by a series of n identical linear reservoirs with uncertain reservoir constant k. Using the continuity equation and the storage-discharge rating equation for a linear reservoir, the outflow from reservoir i can be written

$$q_i = q_{i-1} - k\frac{dq_i}{dt} \quad \text{(2)}$$

in which q_i = outflow rate from reservoir i, q_{i-1} = the input rate (and outflow) from reservoir $i - 1$, and k = reservoir constant. Expressing the parameter k as its mean plus a fluctuation about the mean ($k = \bar{k} + k'$, in which $\bar{k}$ = mean storage coefficient and k' = the zero-mean random fluctuation) results in

$$q_i = q_{i-1} - (\bar{k} + k')\frac{dq_i}{dt} \quad \text{(3)}$$

To develop an instantaneous unit hydrograph the input to the first reservoir should be a Dirac delta function, $\delta(t)$. Thus the outflow, q_1, from the first reservoir is governed by

$$q_1 = \delta(t) - (\bar{k} + k')\frac{dq_1}{dt} \quad \text{(4)}$$

Dividing by $\bar{k}$ and rearranging terms yields

$$\frac{dq_1}{dt} + \frac{q_1}{\bar{k}} = \frac{\delta(t)}{\bar{k}} - \frac{k'}{\bar{k}}\frac{dq_1}{dt} \quad \text{(5)}$$

In this form it is appropriate to seek a solution using Green's functions. The appropriate Green's function is given (Dettman 1962; Bender and Orszag 1978) as the solution to

$$\frac{dG}{dt} + \frac{1}{\bar{k}}G = \delta(t - s) \quad \text{(6)}$$

The result is

$$G = e^{(t-s)/\bar{k}} \quad \text{(7)}$$

The solution to (5) is then given by (Dettman 1962; Bender and Orszag 1978)

$$q_1 = \frac{1}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \delta(s)\, ds - \frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{dq_1}{ds}\, ds \quad \text{(8)}$$

Integration of the first term on the right is straightforward and leads to the partial solution

$$q_1 = \frac{1}{\bar{k}} e^{-t/\bar{k}} - \frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{dq_1}{ds}\, ds \quad \text{(9)}$$

Integration of the second term is more difficult as it contains the q_1. To alleviate this difficulty, one can expand the q_1 as an infinite series (Adomian 1983; Serrano 1988; Sarino and Serrano 1990)

$$q_1 = \xi_0' + \xi_1' + \xi_2' + \xi_3' + \cdots \quad \text{(10)}$$

According to Adomian (1983), this series can be considered a decomposition of q. That is, the approximation assumes that the function is decomposed into a sum of impulses, the same assumption used in finding Green's functions.

Making the substitution of (10) into (8) leads to

$$q_1 = \xi_0' - \frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{d\xi_0'}{ds}\, ds - \frac{K'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{d\xi_1'}{ds}\, ds - \cdots \quad \text{(11)}$$

The components, ξ_i, are determined recursively. Adomian (1983) takes the first term to be the solution to the deterministic differential equation. That solution is the first term on the right of (9). Thus

$$\xi_0' = \frac{1}{\bar{k}} e^{-s/\bar{k}} \quad \text{(12)}$$

and in general

$$\xi_i' = -\frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{d\xi_{i-1}'}{ds}\, ds \qquad (i = 1, 2, \ldots) \quad \text{(13)}$$

The solution for a single reservoir is given by

$$q_1 = e^{-t/\bar{k}} \left[\frac{1}{\bar{k}} + \frac{K't}{\bar{k}^3} + \frac{(K')^2 t(t - 2\bar{k})}{2\bar{k}^5} + \frac{(k')^3 t(6\bar{k}^2 - 6\bar{k}t + t^2)}{6\bar{k}^7} + \cdots \right] \quad \text{(14)}$$

which agrees with the result of Sarino and Serrano (1990).

A similar procedure must be applied to the second reservoir of the cascade. The basic equation is

$$q_2 = q_1 - (\bar{k} + k') \frac{dq_2}{dt} \quad \text{(15)}$$

where q_1 is given by the solution for the first reservoir in (14). Rearranging (15) gives

$$\frac{dq_2}{dt} + \frac{1}{\bar{k}} q_2 = \frac{1}{\bar{k}} q_1 - \frac{k'}{\bar{k}} \frac{dq_2}{dt} \quad \text{(16)}$$

Again using the Green's function, (16) becomes

$$q_2 = \frac{1}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} q_1 \, ds - \frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{dq_2}{ds} \, ds \quad \text{(17)}$$

The first term on the right is easily integrated whereas the second term contains the q_2. As in the solution for (9), q_2 can be decomposed into

$$q_2 = \zeta_0' + \zeta_1' + \zeta_2' + \cdots \quad \text{(18)}$$

where the terms are given by

$$\zeta_0' = \frac{1}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} q_1(s) \, ds \quad \text{(19)}$$

$$\zeta_i' = -\frac{k'}{\bar{k}} \int_0^t e^{-(t-s)/\bar{k}} \frac{d\zeta_{i-1}'}{ds} \, ds \qquad (i = 1, 2, \ldots) \quad \text{(20)}$$

Integrating (19) and (20) recursively leads to the solution for the second reservoir

$$q_2 = te^{-t/\bar{k}} \left[\frac{1}{\bar{k}^2} - \frac{K'(\bar{k} - t)}{\bar{k}^4} + \frac{(k')^2(2\bar{k}^2 - 4\bar{k}t + t^2)}{2\bar{k}^6} - \frac{(k')^2(6\bar{k}^3 - 18\bar{k}^2 t + 9\bar{k}t^2 - t^3)}{6\bar{k}^8} + \cdots \right] \quad \text{(21)}$$

This too agrees with the results of Sarino and Serrano (1990) except for a typographical sign error in their published solution.

Solutions for additional reservoirs in the cascade follow the same pattern. Solutions for the third and fourth reservoir are

$$q_3 = t^2 e^{-t/\bar{k}} \left[\frac{1}{2\bar{k}^3} - \frac{k(2\bar{k} - t)}{2\bar{k}^5} + \frac{(k')^2(6\bar{k}^2 - 6\bar{k}t + t^2)}{4\bar{k}^7} - \frac{(k')^3(24\bar{k}^3 - 36\bar{k}^2 t + 12\bar{k}t^2 - t^3)}{12\bar{k}^9} + \cdots \right] \quad \text{(22)}$$

$$q_4 = t^3 e^{-t/\bar{k}} \left[\frac{1}{6\bar{k}^4} - \frac{k'(3\bar{k} - t)}{6\bar{k}^6} + \frac{(k')^2(12\bar{k}^2 - 8\bar{k}t + t^2)}{12\bar{k}^8} - \frac{(k')(60\bar{k}^3 - 60\bar{k}^2 t + 15\bar{k}t^2 - t^3)}{36\bar{k}^{10}} + \cdots \right] \quad \text{(23)}$$

Using the pattern developing through (14), (21), (22), and (23), a general solution for n reservoirs is inferred to be given by

$$q_n = t^{n-1} e^{-t/\bar{k}} \left\{ \frac{1}{(n-1)!\bar{k}^n} - \frac{K'[(n-1)\bar{k} - t]}{(n-1)!\bar{k}^{n+2}} + \frac{(k')^2[n(n-1)\bar{k}^2 - 2n\bar{k}t + t^2]}{2(n-1)!\bar{k}^{n+4}} - \frac{(K')^3[(n+1)n(n-1)\bar{k}^3 - 3(n+1)n\bar{k}^2 t + 3(n+1)\bar{k}t^2 - t^3]}{6(n-1)!\bar{k}^{n+6}} + \cdots \right\} \quad \text{(24)}$$

The general convergence of the decomposition series approach has been investigated by Adomian (1983) and Serrano (1988). Sarino and Serrano (1990) investigated the number of terms necessary for their particular example and showed that three were adequate. We will also show that three terms are adequate for our case.

A more useful representation of this unit hydrograph is obtained by determining the mean of (24). Assuming that k' is a Gaussian random variable, taking expectations on both sides of (24) and neglecting the very small terms lead to the mean function for random instantaneous unit hydrograph

$$E(q_n) = t^{n-1}e^{-t/\bar{k}}\left\{\frac{1}{(n-1)!\bar{k}^n} + \frac{\sigma^2[n(n-1)\bar{k}^2 - 2n\bar{k}t + t^2]}{2(n-1)!\bar{k}^{n+4}}\right\} \quad \ldots (25)$$

where $E\{\ \}$ denotes the expectation operator and σ = the variance of k'.

COMPARISON WITH OBSERVED HYDROGRAPHS

The stochastic unit hydrograph, (25), was applied to Goodwater Creek, a small watershed located in central Missouri. This 12.2 km^2 agricultural watershed was established as a research catchment by the U.S. Department of Agriculture, Agricultural Research Service in 1971. Rainfall is measured using an array of three recording raingages. The runoff is measured using a concrete broad-crested weir that provides control for low flows whereas a bridge opening provides control for high flows. Twenty-four of the largest observed rainfall-runoff events were selected for comparison. A Gamma function unit hydrograph was determined for each of these events. The base-flow was removed from the observed runoff using a straight line. The fitting parameters, n and k, for the Gamma function unit hydrograph were determined using the Hooke-Jeeves (1961) algorithm, which gives a least-squares best fit. A weighting system was used to emphasize the fitting near the peak discharge. The weight was the ratio of the observed discharge at the time to the peak discharge for the event. Unit hydrographs for time intervals of 1 hour, 30 min, and 15 min were derived. There was a slight difference between the 1-hour and the 30-min unit hydrographs, but no significant difference betweenthe 30-min and the 15-min unit hydrographs. Thus, the 30-min unit hydrograph was selected as representative of the instantaneous unit hydrograph.

The results of fitting the Gamma function to the observed data are listed in Table 1. The tabulated root-mean-square error (column labeled RMS) includes the weighting factor. The average value of n is 3.20, and the average for k is 1.87 hours. The variance for k is 0.44. The results demonstrate the variability typically found when unit hydrographs are determined from several storms. The Gamma function unit hydrograph using these average values will be called the "Gamma using simple means." Note that the coefficient of variation for n is 0.28 and for k is 0.24. Thus the uncertainty in n is similar to the uncertainty in the magnitude of k. In the stochastic model developed here the focus is limited to uncertainty in k. This data set, then, is not ideal for testing the stochastic model.

The stochastic unit hydrography (24) was developed based upon the assumption of an integer number of reservoirs. To test the stochastic version, Gamma functions were fit to the observed rainfall-runoff data with fixed, integer values for n. The ability to fix the value of n and fit the function by optimizing on the parameter k is an advantage of using the Hooke-Jeeves

TABLE 1. Gamma Function Unit Hydrograph Parameters Determined by Fitting 24 Events Observed on Goodwater Creek, North Central Missouri

Flood (1)	Flood date (2)	n (3)	k (hours) (4)	Error (rms) (5)
1	May 6, 1977	3.77	1.15	10.06
2	May 30, 1974	1.53	2.87	32.52
3	September 23, 1986	2.82	1.41	18.46
4	July 3, 1980	2.47	1.93	28.73
5	May 16, 1986	6.41	0.92	10.54
6	April 30, 1983	3.31	1.60	28.90
7	March 3, 1976	3.35	1.49	21.30
8	June 8, 1974	2.95	1.78	25.13
9	June 19, 1983	4.07	1.41	32.81
10	July 24, 1981	3.21	1.45	63.14
11	July 23, 1981	2.98	1.92	58.22
12	November 19, 1985	3.58	1.61	24.56
13	September 1, 1982	2.88	2.00	64.20
14	December 2, 1982	2.84	1.95	85.56
15	April 10, 1979	2.96	1.93	23.83
16	June 19, 1981	2.57	2.09	199.30
17	June 16, 1985	2.37	2.31	55.30
18	October 31, 1984	3.23	1.94	30.05
19	November 18, 1985	2.84	2.09	25.19
20	August 1, 1978	2.91	2.29	38.60
21	June 8, 1984	2.89	2.36	58.81
22	August 29, 1982	4.45	1.78	26.53
23	June 8, 1982	2.55	2.59	21.98
24	August 26, 1982	3.74	1.97	33.04
[Average values]	—	3.20	1.87	42.37
[Standard deviation]	—	0.89	0.44	—
[Coefficient variation]	—	0.28	0.24	—

approach. The values of $n = 2$, 3, and 4 were selected and the results shown in Table 2. As one would expect, the best fit, in the least-squares sense, was with $n = 3$, close to the mean value for fitting with variable n.

Using the best value for n ($n = 3$) and a statistical representation for k (mean = 1.89, variance = 0.31), one can compare the accuracy of the expected unit hydrograph (25) to the "Gamma using simple means" with n and k from Table 1 ($n = 3.2$, $k = 1.87$). The results of this comparison are shown in Table 3. Note that in this table the root-mean-square error is not weighted. The average error for the stochastic unit hydrograph is 59.91, which is slightly smaller than the 60.89 error for the "Gamma using simple means."

EXTENDED STOCHASTIC UNIT HYDROGRAPH

The factorial and the Gamma function are closely related. This permits writing (24) in the form

$$q_n = t^{n-1}e^{-t/\bar{k}}\left\{\frac{1}{\Gamma(n)\bar{k}^n} - \frac{k'[(n-1)\bar{k} - t]}{\Gamma(n)\bar{k}^{n+2}}\right.$$

TABLE 2. Results of Fitting Gamma Function Unit Hydrograph with Fixed Integer n to 24 Events Observed on Goodwater Creek

Flood (1)	Flood date (2)	$n = 2$ k (3)	$n = 2$ Error (4)	$n = 3$ k (5)	$n = 3$ Error (6)	$n = 4$ k (7)	$n = 4$ Error (8)
1	May 6, 1977	2.13	34.78	1.43	15.83	1.08	10.37
2	May 30, 1974	2.18	38.48	1.43	70.73	1.07	95.43
3	September 23, 1986	1.99	40.56	1.32	17.36	0.99	29.54
4	July 3, 1980	2.39	37.77	1.58	28.22	1.17	41.60
5	May 16, 1986	2.93	38.92	1.96	25.30	1.48	16.92
6	April 30, 1983	2.62	76.81	1.76	29.86	1.32	44.27
7	March 3, 1976	2.52	39.37	1.67	22.15	1.25	24.87
8	June 8, 1974	2.62	52.04	1.75	25.80	1.31	35.16
9	June 19, 1983	2.86	72.30	1.91	42.68	1.44	32.82
10	July 24, 1981	2.41	63.66	1.55	63.56	1.16	66.41
11	July 23, 1981	2.83	95.38	1.91	58.46	1.45	89.22
12	November 19, 1985	2.87	69.31	1.92	31.22	1.45	27.95
13	September 1, 1982	2.84	139.21	1.92	64.27	1.45	114.97
14	December 2, 1982	2.76	107.89	1.85	85.95	1.39	93.33
15	April 10, 1979	2.87	46.83	1.90	24.13	1.42	44.39
16	June 19, 1981	2.67	221.22	1.80	199.31	1.37	214.63
17	June 16, 1985	2.76	79.57	1.81	69.71	1.35	130.22
18	October 31, 1984	3.12	87.86	2.09	31.91	1.57	49.84
19	November 18, 1985	3.01	40.50	1.98	25.39	1.47	29.22
20	August 1, 1978	3.36	50.46	2.22	39.72	1.67	60.58
21	June 8, 1984	3.43	74.25	2.27	60.53	1.71	87.98
22	August 29, 1982	3.92	62.89	2.63	36.09	1.98	26.70
23	June 8, 1982	3.34	32.72	2.19	29.35	1.65	54.73
24	August 26, 1982	3.62	55.98	2.44	33.53	1.84	36.18
[Average values]	—	2.84	69.12	1.89	47.13	1.42	60.72
[Standard deviation]	—	0.46	—	0.31	—	0.24	—

$$+ \frac{(k')^2[n(n-1)\bar{k}^2 - 2n\bar{k}t + t^2]}{2\Gamma(n)\bar{k}^{n+4}}$$

$$\left. - \frac{(k')^3[(n+1)n(n-1)\bar{k}^3 - 3(n+1)n\bar{k}^2 t + 3(n+1)\bar{k}t^2 - t^3]}{6\Gamma(n)\bar{k}^{(n+6)}} \right\} \quad \ldots\ldots (26)$$

This equation can be used for n that is not an integer. Thus, although not directly derived for a fractional number of reservoirs, (26) can be applied to such a case so that it may fit observed data more precisely. Similarly, the equation for the mean stochastic unit hydrograph (25) can be expressed in terms of the Gamma function with the result

$$E(q_n) = t^{n-1}e^{-t/\bar{k}} \left\{ \frac{1}{\Gamma(n)\bar{k}^n} + \frac{\sigma^2[n(n-1)\bar{k}^2 - 2n\bar{k}t + t^2]}{2\Gamma(n)\bar{k}^{n+4}} \right\} \quad \ldots\ldots\ldots (27)$$

where $E\{\,\}$ = the expectation operator and σ^2 = the variance of k'. If the parameter K' is constant for different storm events, that is $\sigma = 0$, (27) will reduce to (1), the deterministic Gamma function unit hydrograph.

TABLE 3. Comparison of Stochastic Unit Hydrograph (n = 3, k_{mean} = 1.98, $k_{variance}$ = 0.31) with "Gamma Using Simple Means" of Table 1

Flood (1)	Flood date (2)	Error (rms): Gamma using simple means (3)	Error (rms): Stochastic (n = 3) (4)
1	May 6, 1977	50.39	43.87
2	May 30, 1974	113.60	96.93
3	September 23, 1986	77.84	67.18
4	July 3, 1980	52.14	43.01
5	May 16, 1986	22.86	28.30
6	April 30, 1983	60.04	47.58
7	March 3, 1976	41.63	35.80
8	June 8, 1974	36.24	29.83
9	June 19, 1983	42.18	47.13
10	July 24, 1981	59.18	58.96
11	July 23, 1981	62.17	58.04
12	November 19, 1985	32.39	37.47
13	September 1, 1982	65.95	61.93
14	December 2, 1982	82.20	83.77
15	April 10, 1979	31.51	25.73
16	June 19, 1981	174.53	180.83
17	June 16, 1985	102.04	73.23
18	October 31, 1984	35.36	57.40
19	November 18, 1985	25.62	28.13
20	August 1, 1978	49.88	56.67
21	June 8, 1984	75.79	85.91
22	August 29, 1982	71.75	83.25
23	June 8, 1982	36.85	37.34
24	August 26, 1982	59.27	69.53
[Average values]	—	60.89	59.91

This extended result can also be compared with the "Gamma using simple means" for n and k. The results are given in Table 4. Again, the stochastic unit hydrograph is slightly better than the "Gamma using simple means." The root-mean-square error term for the integer value of n (n = 3) is slightly better than that for the noninteger value (n = 3.2). We attribute this to the shift from using a weighted root-mean-square error to determine the coefficients to an unweighted root-mean-square error in this test. The results for all tests are quite close.

CONVERGENCE OF DECOMPOSITION SERIES

The convergence of the decomposition series can be tested using the fitted values of n (n = 3.20) and a statistical representation for k (mean = 1.87, variance = 0.44). The number of terms necessary in the decomposition series when used in this particular case will be established. A realization of the random component of k equal to the variance of k' allows a comparison of the magnitude of the terms containing a different power of k'. The results are illustrated in Fig. 1. The value of the terms, q_1'', q_2'', q_3'', and q_4'', in the

TABLE 4. Comparison of Extended Stochastic Unit Hydrograph (n = 3.2, k_{mean} = 1.87, $k_{variance}$ = 0.31) with "Gamma Using Simple Means" of Table 1

Flood (1)	Flood date (2)	Error (rms): Gamma using simple means (3)	Error (rms): Stochastic (n = 3.2) (4)
1	May 6, 1977	50.39	42.21
2	May 30, 1974	113.60	111.30
3	September 23, 1986	77.84	76.89
4	July 3, 1980	52.14	53.58
5	May 16, 1986	22.86	27.44
6	April 30, 1983	60.04	60.37
7	March 3, 1976	41.63	44.57
8	June 8, 1974	36.24	39.04
9	June 19, 1983	42.18	51.90
10	July 24, 1981	59.18	60.86
11	July 23, 1981	62.17	64.94
12	November 19, 1985	32.39	36.56
13	September 1, 1982	65.95	55.68
14	December 2, 1982	82.20	85.02
15	April 10, 1979	31.51	34.49
16	June 19, 1981	174.53	174.46
17	June 16, 1985	102.04	79.97
18	October 31, 1984	35.36	44.70
19	November 18, 1985	25.62	32.74
20	August 1, 1978	49.88	46.40
21	June 8, 1984	75.79	70.26
22	August 29, 1982	71.75	69.74
23	June 8, 1982	36.85	30.01
24	August 26, 1982	59.27	55.07
[Average values]	—	60.89	60.34

decomposition series $q_n = q_1'' + q_2'' + q_3'' + q_4''$ are shown. These terms are given by

$$q_1'' = t^{n-1}e^{-t/\bar{k}}\left[\frac{1}{\Gamma(n)\bar{k}^n}\right] \quad (28)$$

$$q_2'' = t^{n-1}e^{-t/\bar{k}}\left\{-\frac{k'[(n-1)\bar{k}-t]}{\Gamma(n)\bar{k}^{n+2}}\right\} \quad (29)$$

$$q_3'' = t^{n-1}e^{-t/\bar{k}}\left\{\frac{(k')^2[n(n-1)\bar{k}^2-2n\bar{k}t+t^2]}{2\Gamma(n)\bar{k}^{n+4}}\right\} \quad (30)$$

$$q_4'' = t^{n-1}e^{-t/\bar{k}} \cdot\left\{\frac{(k')^3[(n+1)n(n-1)\bar{k}^3-3(n+1)n\bar{k}^2t+3(n+1)\bar{k}t^2-t^3]}{6\Gamma(n)\bar{k}^{n+6}}\right\} \quad (31)$$

These results indicate that only two terms in the random IUH, that is $q_n = q_1'' + q_2''$, are enough to achieve an acceptable accuracy.

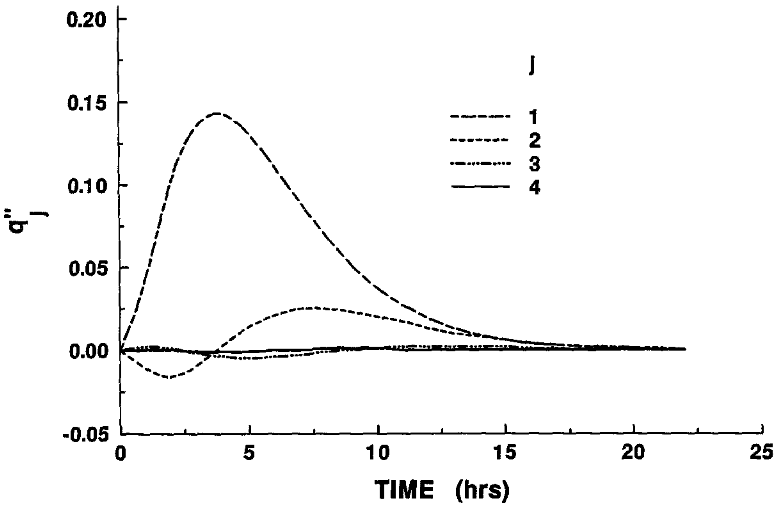

FIG. 1. Magnitude of Terms in Decomposition Series

CONCLUSIONS

Stochastic models can explicitly describe the variability in the properties used to fit models to rainfall and runoff data. A general form for the stochastic unit hydrograph was developed through an application of stochastic differential equations to the conceptual cascade of reservoirs model for the rainfall-runoff process. The method was applied to a small watershed in central Missouri. A test of convergence of the decomposition series used showed that three terms gave adequate results. The results showed a good agreement for the observed runoff hydrographs.

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