Closed-Form Expressions for Water Retention and Conductivity Data

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Abstract

Closed-form expressions for quantifying the unsaturated soil hydraulic properties are widely used in computer programs to model subsurface flow and transport in porous media and to investigate indirect methods for estimating these properties. For example, water retention data, which relate soil-water pressure head (h) and effective water saturation (S_e), are frequently used to predict the unsaturated hydraulic conductivity (K). However, the suitability of different functions to describe unsaturated hydraulic data has rarely been investigated comprehensively. We attempted to fit 14 retention and 11 conductivity functions to 903 sets of water retention and hydraulic conductivity data measured on soil and rock samples or horizons reported in the unsaturated hydraulic database UNSODA. Some of the best mean values for r^2 and MSE for fitting $S_e(h)$ data were obtained with the retention functions reported by van Genuchten (1980), Globus (1987), and Hutson and Cass (1987). A function reported by Gardner (1958) could describe K(h) data quite well whereas functions reported by Brooks and Corey (1964) and van Genuchten (1980), which are respectively based on the conductivity models by Burdine and Mualem, yielded a relatively good description of $K(S_e)$ data.

Introduction

Computer models are now routinely used to simulate water flow and chemical transport in the vadose zone to better understand and manage contamination of the subsurface environment or for optimizing agricultural production. The accurate solution of the unsaturated flow problem, which was first formulated by Richards (1931), is a fundamental step in such numerical simulations. A considerable amount of work has been devoted to quantify the unsaturated soil hydraulic properties, i.e., the water retention curve, which relates effective water saturation (S_e) and matric head (h), and the hydraulic conductivity curve, which gives the hydraulic conductivity (K) as a function of water saturation or pressure head. The terminology in the literature may be different from ours, for example, soil-water characteristic instead of water retention curve and potential for head. We will assume that $h = 0$ for saturated conditions and $h > 0$ for unsaturated conditions.

Closed-form expressions have been widely employed to describe the unsaturated hydraulic properties for many reasons (van Genuchten and Leij, 1991; Marion et al., 1994). They are attractive to model the $S_e(h)$ and K(h) or $K(S_e)$ relationships in numerical models; the nontabular data will simplify input to computer models (although at the expense of computational efficiency) and allow the estimation of hydraulic properties using inverse procedures. Analytical expressions facilitate rapid comparison of hydraulic properties of different porous media. They are used to estimate hydraulic properties for media containing different types of fluids, they are convenient for scaling approaches or for use with Geographic Information Systems, and they may be used to estimate hydraulic properties in the dry range. Closed-form expressions play an important role in *indirectly* quantifying unsaturated hydraulic data using soil properties that are already available or can easily be determined. Parameters in the analytical expressions for $S_e(h)$, K(h), and $K(S_e)$ can be estimated empirically with regression equations that correlate textural and other data with unsaturated soil hydraulic data. Inasmuch as the measurement of the hydraulic conductivity is considerably more difficult and less accurate than that of the water retention curve, a number of physico-empirical approaches have been proposed to estimate K(h) or $K(S_e)$ from a closed-form expression for $S_e(h)$ data.

Because of the aforementioned utility and convenience, analytical expressions have become quite popular for describing hydraulic properties. Many functions have been proposed for this purpose. Regardless of their application, it is imperative that these functions accurately describe the unsaturated soil hydraulic data. A limited number of independent studies have been published that evaluate the suitability of several popular empirical and semiempirical functions. For example, Alexander and Skaggs (1986) investigated the prediction of conductivity data from tabular or functional retention data using 23 data sets. The best description was apparently obtained by using the retention function of Campbell (1974) and an expression for K(h) presented by Alexander and Skaggs (1986). Van Genuchten and Nielsen (1985) investigated four versions of the retention function of van Genuchten (1980) with slope parameter n (β according to our notation) and transformation parameter m (γ in this paper). In one version the parameter γ is flexible, in two versions γ is respectively given by $\gamma = 1 - 1/\beta$ and $\gamma = 1 - 2/\beta$ to facilitate use of the conductivity functions by Mualem (1976a) and Burdine (1953), and in a fourth version $\gamma \to \infty$ to obtain a function according to Brooks and Corey (1964). A set of 102 retention curves documented by Mualem (1976b) was generally best described when γ was flexible while the restrictions $\gamma = 1 - 1/\beta$, $\gamma = 1 - 2/\beta$, and $\gamma \to \infty$ resulted, on average, in a progressively worse fit. However, van Genuchten and Nielsen (1985) did not analyze sufficient conductivity data sets to judge the accuracy of different conductivity functions. Although closed-form expressions for the unsaturated hydraulic properties are widely employed, it appears that relatively little attention is being paid in the literature to a systematic evaluation of such functions on a large number of data sets containing observations for water retention *and* unsaturated hydraulic conductivity.

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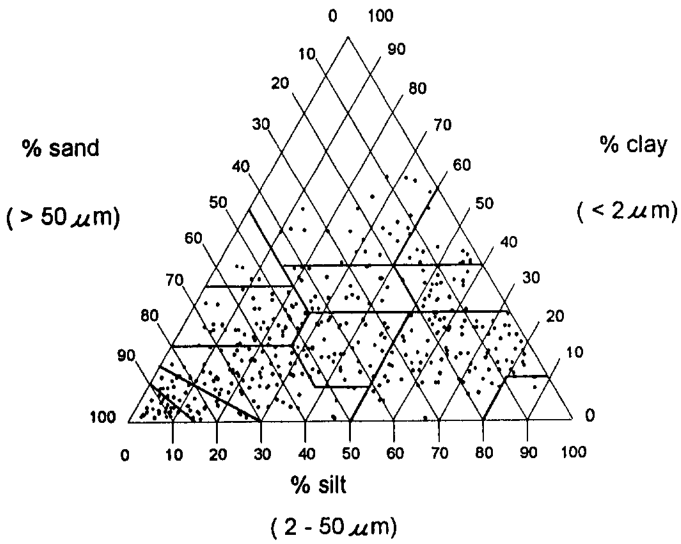

Fig. 1. Distribution of data sets across the textural triangle.

The objective of this paper is to evaluate a variety of functional expressions for $S_e(h)$ and $K(h)$ or $K(S_e)$ to describe the hydraulic data and other relevant information contained in the international UNsaturated SOil hydraulic DAtabase (Leij et al., 1996). This database contains data sets from around the world for a variety of porous media; it may be obtained electronically (http://www.epa.gov/ada/kerrlab.html). Figure 1 presents the distribution of the samples in UNSODA in a textural triangle showing the percentages of clay (< 2 µm), silt (2-50 µm), and sand (> 50 µm). It is apparent that a large fraction of the samples in UNSODA are sands and sandy loams.

Closed-Form Expressions

Closed-form expressions were selected after a review of the literature and a preliminary screening of retention and conductivity functions. We only selected models for which we believed that the parameter estimation would converge for a majority of data sets. Furthermore, we have attempted to assemble different types of mathematical formulations, including expressions based on power functions, polynomials, and exponential and error functions. Additional functions for the hydraulic curves, especially the water retention, can be found in the literature (cf. Globus, 1987; Mualem, 1986; Vereecken, 1992).

Retention Functions

The 14 retention models used for this study are listed in Table 1. Note that $S_e = (\theta - \theta_r)/(\theta_s - \theta_r)$ is an effective, dimen-

Table 1. Expressions for Water Retention, $S_e(h)$

Model	*Expression*		*Reference*
11	$\begin{cases} 1 \\ (\alpha h)^{-\beta} \end{cases}$	$\alpha h \leq 1$ $\alpha h > 1$	Brooks and Corey (1964)
12	$\frac{1}{2}\mathrm{erfc}\left[\gamma - \frac{\beta}{1+\alpha h}\right]$		Laliberte (1969)
13	$\alpha/[\alpha + (\ln h)^{\beta}]$	$(h \geq 1)$	Vauclin et al. (1979)
14	$[1 + (\alpha h)^{\beta}]^{-\gamma}$		van Genuchten (1980)
15	$[1 + (\alpha h)^{\beta}]^{(1-\beta)/\beta}$		van Genuchten (1980)
16	$\begin{cases} 1 - (\alpha h)^2 S_i^{2/\beta}(1 - S_i) \\ (\alpha h)^{-\beta} \end{cases}$	$h \leq h_i$ $h > h_i$	Hutson and Cass (1987)
	where $h_i = 1/\alpha S_i^{1/\beta}$ and $S_i = 2/(2 + \beta)$		
17	$[(1 + \frac{1}{2}\alpha h)\exp(-\frac{1}{2}\alpha h)]^{1/(1+\beta)}$		Russo (1988)
18	$\frac{1 + \gamma\alpha h}{1 + \alpha h + \beta(\alpha h)^2}$		Zhang and van Genuchten (1994)
19	$\frac{1 + (\alpha\beta)^{\gamma}}{1 + [\alpha(h + \beta)]^{\gamma}}$		Globus (1987)
20	$1 - \beta\ln(\alpha h)$		Farrell and Larson (1972)
21	$1 - b(\alpha h)^2$ $(\alpha h)^{-\gamma} - (\alpha h_d)^{-\gamma} + a\ln(h_d/h)$	$0 < h < \beta$ $h > \beta$	Rossi and Nimmo (1994)
	$a = \frac{2 - (2 + \gamma)(\alpha\beta)^{\gamma} + 2(\alpha h_d)^{-\gamma}}{1 + 2\ln(h_d/\beta)}$		
	$b = \left[\frac{\gamma}{(\alpha\beta)^{\gamma}} + a\right] / [2(\alpha\beta)^2]$		
22	$\begin{cases} \frac{1}{2}\mathrm{erfc}\left(\frac{\ln[(1 - \alpha h)/(1 - \alpha\beta)] - \gamma^2}{\gamma(2)^{1/2}}\right) \\ 1 \end{cases}$	$\alpha h > 1$ $\alpha h \leq 1$	Kosugi (1994)
23	$\exp[\beta(1 - \alpha h)]$		Bumb et al. (1992)
24	$\{1 + \exp[\beta(\alpha h - 1)]\}^{-1}$		Bumb et al. (1992)

Table 2. Expressions for Relative Hydraulic Conductivity, K/K_s

Function	*Expression*	*Reference*
h based		
31	$[1+(\alpha h)^{\delta}]^{-\gamma}$	Gardner (1958)
32	$\exp(-\lvert\delta h\rvert)$	Gardner (1958)
33	$\begin{cases} 1 & \alpha h < 1 \\ (\alpha h)^{-2-3\delta} & \alpha h > 1 \end{cases}$	Brooks and Corey (1964)
34[a]	$S_e^{\delta}\left\{ \int_0^S \frac{dx}{h(x)} \Big/ \int_0^1 \frac{dx}{h(x)} \right\}^2$	Mualem (1976)
S_e based		
41	$S_e^{3+2/\delta}$	Brooks and Corey (1964)
42	$S_e^{\delta}[1-(1-S_e^{1/\gamma})^{\gamma}]^2$	van Genuchten (1980)
43	$S_e^{\delta}[I_{\zeta}(p,q)]^2 \quad (p=\gamma+1/\beta;\ q=1-1/\beta;\ \zeta=S_e^{1/\gamma})$	van Genuchten and Nielsen (1985)
44	$\left(\frac{S_e}{\delta}\right)^{0.5}\left(\frac{1-(1-S_e^{1/\gamma})^{\gamma}}{1-(1-\delta^{1/\gamma})^{\gamma}}\right)^2$	van Genuchten et al. (1991)
45	$\frac{(1-\gamma)\,S_e^{\delta}}{1-S_e}$	Fujita (1952)
46	$\exp(-\gamma S_e^{\delta})$	Setiawan and Nakano (1993)
47	$\exp[-\gamma(S_e-\delta)]$	Libardi et al. (1980)

[a]cf. Table 3.

sionless reduced water content, and θ is the water content expressed as volume of water per volume of porous medium. The subscripts r and s denote the residual and saturated water contents, respectively. The retention functions may also contain up to three empirical shape factors α, β, and γ. The model by Brooks and Corey (1964), which is given by function 11, has been successfully used to describe retention data for relatively homogeneous and isotropic samples, which have a narrow pore-size distribution, with a typical value for β of 2 (lower for structured soils and higher for sands). Parameter values can be readily obtained from a plot of the $S_e(h)$ curve on double logarithmic paper. Function 11 may not describe the data well near saturation where the saturation is fixed and a discontinuity occurs at $h = 1/\alpha$. The same problem plagues the function by Campbell (1974), which has a similar form as 11 with a reverse dependency of pressure head and water content. Several functions have been suggested to improve the description near saturation.

Function 12 is a variation of the function presented by Laliberte (1969). It is intended to provide a smooth description of the data near saturation with a continuous derivative. We investigated function 12 without imposing any conditions on α, β, and γ. Retention function 13 of Vauclin et al. (1979) has been used by El-Kadi (1985); its form is similar to the expression proposed by Brutsaert (1966) except that a log-transformed pressure head is being used. Note that $S_e = 1$ for $0 \le h < 1$. Function 14 of van Genuchten (1980) has been widely used for describing retention data. This function allows for a smooth transition zone near $h = 1/\alpha$ in contrast with function 11 (see also conductivity function 31). A similar function, with $\gamma = 1$, has been used by Ahuja and Swartzendruber (1972) and Varallyay and Mironenko (1979). To facilitate the use of function 14 in hydraulic conductivity models, the value of γ is sometimes restricted. Function 15 involves the restriction $\gamma = 1 - 1/\beta$, which is used in conjunction with the conductivity model by Mualem (1976a).

The previously mentioned function by Campbell (1974) was modified by Clapp and Hornberger (1978) by describing h as a function of $S = \theta/\theta_s$ with a parabolic equation close to saturation ($S_i < S \le 1$) in addition to the power function of Campbell for $S \le S_i$. Continuity was required in h and dh/dS at S_i, which was arbitrarily set equal to 0.92. Hutson and Cass (1987) employed this concept by using a parabolic equation that did not lead to additional unknown parameters while allowing for a flexible S_i. Function 16 gives their two-part expression for the water retention and the definition for S_i.

Russo (1988) derived retention function 17 for use with conductivity function 32 of Gardner (1958), shown in Table 2, using the constraint that the retention function satisfies the capillary model for predicting conductivity by Mualem (1976a). Zhang and van Genuchten (1994) proposed function 18 to describe retention data for porous media with a bimodal pore-size distribution. This function has a relatively simple form as it is the ratio of a first- and second-order polynomial. Retention curves of media with a multimodal pore-size distribution have also been described by summing functions of the same form, such as function 14 (cf. Othmer et al., 1991; Durner, 1994) or of a different form (Ross and Smettem, 1993). Retention function 19, which was reported in the monograph by Globus (1987), is similar to function 18 except that there is no independent variable in the numerator. Farrell and Larson (1972) arbitrarily assumed a linear relationship between h and $\exp(\theta_s - \theta)$; function 20 is obtained by changing the dependency.

Several recent publications present retention equations that were modified to improve the description of water retention for dry conditions (Ross et al., 1991; Rossi and Nimmo, 1994; Fayer and Simmons, 1995). Function 21 serves as an example of these expressions. This function consists of a power function for the wet range similar to function 16, and the Brooks-Corey function 11 with a logarithmic correction term according to Ross et al. (1991) for the dry range. The saturation and its derivative are continuous at the junction point $h = \beta$.

Kosugi (1994) derived retention function 22 by assuming a modified three-parameter lognormal expression for the pore-size probability density function. The maximum pore size of the medium determines the pressure $1/\alpha$ below which the water content is constant ($S_e = 1$). At the inflection point, $h = \beta$, the value for S_e is equal to $\frac{1}{2}\,\mathrm{erfc}[-\gamma/(2)^{1/2}]$ where erfc denotes the complementary error function.

Finally, Bumb et al. (1992) evaluated expressions 23 and 24 for fitting water retention data. Both contain exponential functions and were identified as the Boltzmann and Fermi distribution, respectively. The Boltzmann distribution is convenient for deriving expressions of the hydraulic conductivity because it can be easily inverted to the $h(S_e)$ form and integrated. However, it does not provide a smooth description of the water content for low pressures like the Fermi distribution.

Conductivity Functions

Many general models for the hydraulic conductivity conceptualize flow in the microscopic pore space and use Darcy's law to estimate the (macroscopic) hydraulic conductivity. Explicit functions for the hydraulic conductivity can be defined from these models by specifying a water retention curve, which is expressed in functional form as $h(S_e)$, to estimate the pore-size distribution. Two of the more popular models were proposed by Burdine (1953):

$$K(S_e) = K_s S_e^{\delta} \frac{g(S_e)}{g(1)} \quad \text{where} \quad g(S_e) = \int_0^{S_e} \frac{d\xi}{[h(\xi)]^2} \qquad (1)$$

and Mualem (1976a)

$$K(S_e) = K_s S_e^{\delta} \left[\frac{f(S_e)}{f(1)} \right]^2 \quad \text{where} \quad f(S_e) = \int_0^{S_e} \frac{d\xi}{h(\xi)} \qquad (2)$$

with ξ as a dummy integration variable. Our description of conductivity data typically involves the use of two additional parameters. The fourth empirical constant δ, which is frequently set to 2 for the Burdine model and 0.5 for the Mualem model, and the saturated conductivity, K_s. Note that these types of conductivity models are not applicable for flow in very dry media.

Table 2 contains the functions that were used to describe the relative conductivity, K/K_s. These expressions are either completely empirical or based on a pore-size distribution model. A distinction is made between conductivity functions based on the pressure head, i.e., $K(h)$, and those based on the water content, i.e., $K(S_e)$. The soil-water diffusivity, D, which is sometimes used in the Richards equation instead of the conductivity, may be derived from the conductivity and retention functions according to $D = -K(dh/d\theta)$.

Gardner (1958) derived analytical solutions for steady one-dimensional unsaturated flow assuming two simplified expressions for $K(h)$. Function 31 is a generalization of the function by Gardner, who assumed $\gamma = 1$, to obtain an expression similar to retention function 14. Gardner (1958) reported that function 31 ($\gamma = 1$) seems to fit conductivity data quite well. The analytical solution of unsaturated flow problems may be facilitated if function 32 is used. Gardner (1958) pointed out that this function does not provide a good description of K over a wide range of h. Function 33 is obtained by first evaluating the Burdine model (1) for the Brooks and Corey retention function 11 and then expressing the conductivity in terms of h instead of S_e with the help of function 11.

Function 34 is based on the conductivity model of Mualem (1976a) given by (2) for a retention function that may be selected from 12 expressions in Table 1. Application of the chain rule allows a change in integration variable that obviates the need for an explicit inversion from the $S_e(h)$ to the $h(S_e)$ form. We may write (cf. Russo, 1988; Fayer and Simmons, 1995):

$$f(S_e) = \int_0^{S_e} \frac{dS_e}{h} = \int_{\infty}^{h} \frac{1}{h} \frac{dS_e}{dh}\, dh \equiv f(h) \qquad (3)$$

Table 3 contains the expressions for conductivity function 34. The integration in either equation (2) or (3) can sometimes be done analytically and otherwise has to be carried out numerically. We used Gauss-Chebyshev quadrature for the numerical integration whereas some of the steps in the analytical derivation were conducted with a mathematical software package (Wolfram, 1991). The conductivity function based on retention function 14 includes the complete Beta function, $B(p, q)$. The expression derived from retention function 23 is given in terms of the exponential integral, $\mathrm{Ei}(-\alpha\beta)$, with an upper limit of integration $h^* = 1/(10\alpha)$.

The next group of conductivity functions uses S_e as the independent variable. Function 41, which was presented by Brooks and Corey (1964), is readily obtained by substituting function 11 into equation (1). Van Genuchten (1980) derived function 42 by simplifying function 14 to 15 ($\gamma = 1 - 1/\beta$) and by subsequently evaluating Mualem's conductivity model. It should be noted that the corresponding $K(h)$ function in Table 3 follows from inserting function 15 for S_e into 42. Conductivity function 43 of van Genuchten and Nielsen (1985) was derived according to Mualem's model from the less restrictive retention function 14. We have used the notation of these authors for the incomplete beta function, $I_\zeta(p, q)$. Van Genuchten et al. (1991) list a number of disadvantages in using the saturated hydraulic conductivity as a matching point for the hydraulic conductivity. It may be desirable to use a water content below saturation as a reference value. Conductivity function 44 uses a water content $S_e = \delta$ for this purpose. Function 44 gives the ratio of the conductivities according to function 42 at an arbitrary S_e. It should be noted that a similar approach may be taken for other conductivity models than 42. Furthermore $K(S_e = \delta)$ is ideally fixed to represent a reliable observation point.

The following two functions are slight modifications of the original expressions to improve the parameter optimization while using uniform initial estimates. Function 45 is obtained by modifying the expression reported, among others, by Warrick (1995) and attributed to Fujita (1952). Function 46 follows from changing the equation used by Setiawan and Nakano (1993). Finally, an expression similar to function 47 was used by Libardi et al. (1980) for the in situ measurement of the hydraulic conductivity.

Parameter Optimization

We adapted the program RETC of van Genuchten et al. (1991) to estimate the parameters of all the combinations of

Table 3. Expressions for K/K_s or $f(h)$ to Evaluate Conductivity Function 34 in Table 2 Using the Retention Functions of Table 1

Retention function	*Expression*	
11	$\begin{cases} 1 & \alpha h \le 1 \\ (\alpha h)^{-\beta\delta-4} & \alpha h > 1 \end{cases}$	
12	$f(h) = \int_h^\infty \frac{1}{h} \frac{\alpha\beta}{(\pi)^{1/2}} \frac{\exp\{-[1-\beta/(1+\alpha h)]^2\}}{(1+\alpha h)^2} dh$	
13	$f(h) = \int_h^\infty \frac{\alpha\beta}{h^2} \frac{(\ln h)^{\beta-1}}{[\alpha + (\ln h)^\beta]^2} dh$	
14[a]	$\beta^2/\{[(\beta\gamma+1)\,B(p,q)]^2 (\alpha h)^{2+\beta\gamma(\delta+2)}\}$ (small S_e)	
15[a]	$\frac{\{1-(\alpha h)^{\beta\gamma}[1+(\alpha h)^\beta]^{-\gamma}\}^2}{[1+(\alpha h)^\beta]^{\gamma\delta}}$ $(\gamma = 1 - 1/\beta)$	
16	$\begin{cases} \{[1-(\alpha h)^2 S_i^{2/\beta}(1-S_i)\}^\delta \{[f_1(h)+f_2(h_i)]/[f_1(0)+f_2(h_i)]\}^2 \\ [(\alpha h)^{-\beta}]^\delta \{f_2(h)/[f_1(0)+f_2(h_i)]\}^2 \end{cases}$	$h \le h_i$ $h > h_i$
	where	
	$f_1(h) = 2\alpha^2 S_i^{2/\beta}(1-S_i)(h_i - h)$ $f_2(h) = [\beta h^{-(1+\beta)}]/[(1+\beta)\,\alpha^\beta]$	
18	$\begin{cases} 1 \\ \{(1+\gamma\alpha h)/[1+\alpha h+\beta(\alpha h)^2]\}^\delta [f(h)/f(1/\beta)]^2 \end{cases}$	$h < 1/\alpha$ $h > 1/\alpha$
	where	
	$f(h) = \frac{\alpha(\gamma - 1\beta\alpha h)}{1+\alpha h+\beta(\alpha h)^2} + \frac{\alpha(1-2\beta-\gamma)}{(4\beta-1)^{1/2}} \operatorname{arctg}\left(\frac{1+2\beta\alpha h}{(4\beta-1)^{1/2}}\right) + \alpha(\gamma-1)\{\ln(\alpha h) - \tfrac{1}{2}\ln[1+\alpha h+\beta(\alpha h)^2]\}$	
19	$f(h) = \alpha\gamma[1+(\alpha\beta)^\gamma] \int_h^\infty \frac{1}{h} \frac{[\alpha(h+\beta)]^{\gamma-1}}{\{1+[\alpha(h+\beta)]^\gamma\}^2} dh$	
21[b]	$\begin{cases} [1-b(\alpha h)^2]^\delta \{[f_1(h)+f_2(\beta)]/[f_1(0)+f_2(\beta)]\}^2 \\ [(\alpha h)^{-\gamma} - (\alpha h_d)^{-\gamma} + a\ln(h_d/h)]^\delta \{f_2(h)/[f_1(0)+f_2(\beta)]\}^2 \end{cases}$	$0 < h < \beta$ $h > \beta$
	where	
	$f_1(h) = 2\alpha^2 b(\beta - h)$ $f_2(h) = \frac{1}{h}\left[a + \frac{\gamma(\gamma h)^{-\gamma}}{1+\gamma}\right] - \frac{1}{h_d}\left[a + \frac{\gamma(\alpha h_d)^{-\gamma}}{1+\gamma}\right]$	
22[c]	$\left\{\tfrac{1}{2}\operatorname{erfc}\left[\frac{\ln(h/\beta) - \gamma^2}{\gamma(2)^{1/2}}\right]\right\}^\delta \left\{\tfrac{1}{2}\operatorname{erfc}\left[\frac{\ln(h/\beta)}{\gamma(2)^{1/2}}\right]\right\}^2$	
23	$\{\exp[\beta(1-\alpha h)]\}^\delta [f(h)/f(h_*)]^2$	
	where	
	$f(h) = -\alpha\beta\exp(\beta)\,\mathrm{Ei}(-\alpha\beta)$ $h^* = 1/(10\alpha)$	
24	$f(h) = \int_\infty^h \frac{1}{h} \frac{\alpha\beta\exp[\beta(\alpha h - 1)]}{\{1+\exp[\beta(\alpha h-1)]\}^2} dh$	

[a] van Genuchten et al. (1991).
[b] Fayer and Simmons (1995).
[c] Das and Kluitenberg (1995).

retention and conductivity functions. The parameter values were determined by minimizing the objective function according to

$$\min_{\mathbf{b}} O(\mathbf{b}) = \sum_{i=1}^{N} [\theta_i - \hat{\theta}_i(\mathbf{b})]^2 + \sum_{i=1}^{M} \{W[\log K_i - \log\hat{K}_i(\mathbf{b})]\}^2 \quad (4)$$

where θ_i and $\hat{\theta}_i$ are the observed and fitted water contents, respectively; K and $\hat{K}_i$ are the observed and fitted values for the conductivity data; N is the number of retention data; M is the number of conductivity data; and the trial parameter vector, $\mathbf{b} = (\theta_r, \theta_s, \alpha, \beta, \gamma, \delta, K_s)^T$, contains the unknown model parameters to be fitted. The weight W is internally determined according to

$$W = M \sum_{i=1}^{N} \theta_i \Big/ N \sum_{i=1}^{M} \log K_i \quad (5)$$

This weight is intended to minimize bias in the optimization procedure toward the data type with the greater numerical values by normalizing the conductivity data with respect to the retention data. The optimization was done according to the Levenberg-Marquardt method (Marquardt, 1963).

The goodness of fit for the individual retention and conductivity data sets was assessed with the correlation coefficient (r^2)

$$r^2 = 1 - \frac{\sum_{i=1}^{N} (\theta_i - \hat{\theta}_i)^2 + \sum_{i=1}^{M} (\log K_i - \log\hat{K}_i)^2}{\sum_{i=1}^{N} (\theta_i - \bar{\theta})^2 + \sum_{i=1}^{M} (\log K_i - \log\bar{K})^2}$$

$$\bar{\theta} = \frac{1}{N}\sum_{i=1}^{N} \theta_i \qquad \log\bar{K} = \frac{1}{M}\sum_{i=1}^{M} \log K_i \quad (6)$$

and the mean squared error (MSE) is given by

$$\mathrm{MSE} = \frac{\sum_{i=1}^{N} (\theta_i - \hat{\theta}_i)^2 + \sum_{i=1}^{M} (\log K_i - \log\hat{K}_i)^2}{M + N - p} \quad (7)$$

where p is the number of fitting parameters. We have arbitrarily limited the maximum number of fitting parameters to seven but such a limitation is not necessary. The variation in statistical and model parameters, x, is quantified with the standard error of the mean:

$$\mathrm{SE} = \left[\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{(n-1)\,n}\right]^{1/2} \quad (8)$$

where μ_x is the arithmetic mean, and n is the number of data sets. In addition to these mathematical criteria, one should also check whether the fitting parameters provide a physically realistic description of the hydraulic data.

Fitting parameters for the combinations of the 14 retention with the 11 conductivity functions are given in Table 4. The first column of Table 4 shows the number of the retention function according to Table 1 whereas the numbering of the K(h) and $K(S_e)$ functions in the top row corresponds to that in Table 2. Each of the 54 combinations of $S_e(h)$-K(h) functions was fitted to 346 data sets. No results are reported for combinations 1734 and 2034 because of a lack of data sets that could be successfully optimized. Note that the first two digits of the combination number refer to the retention function and the last two to the conductivity function. Similarly, we tried to fit all 98 combinations of $S_e(h)$-$K(S_e)$ functions to 557 data sets. The top and bottom line of each box contain the fitting parameters used in the retention and conductivity function, respectively. Note that θ_r, θ_s, and K_s were always used as initial fitting parameters. This allows a maximum of four additional fitting parameters. In many cases the retention and conductivity functions contain common fitting parameters while in some cases variables were eliminated as fitting parameters by setting them equal to the numerical values shown in Table 4.

As with any nonlinear estimation procedure, the optimization algorithm will perform better if the initial estimates are close to the "true" solution of the inverse problem. Furthermore, constraints on parameter values during the optimization may improve the performance of the algorithm and prevent the occurrence of run time errors. Such constraints may be physically or mathematically based. Because we are not primarily interested in obtaining a unique solution of the inverse problem for each data set and for each model, which will be quite cumbersome in view of the quantity and nonuniformity of the data, we have generally used a consistent set of initial estimates and implemented only a limited number of parameter constraints.

Initial estimates are given in Table 5; these are based on texture according to the results of Carsel and Parrish (1988). Values of θ_r and θ_s were constrained for both the initial estimates and the optimization. If the initial estimate θ_r^{in} is greater than any of the observed water contents it is set to 90% of the lowest water content. In case the initial estimate for the saturated water content, θ_s^{in}, is less than any of the experimental water contents, it is reset to 105% of the maximum observed θ. After one iteration is completed, the optimization is constrained in a somewhat empirical manner. The lower and upper limits on θ_r are 0 and $1.2\theta_r^{in}$, respectively. If these limits are exceeded, θ_r is no longer optimized and its value is fixed at either 0 or θ_r^{in}. Similarly, the lower and upper limits on θ_s are θ_s^{in} and $1.8\theta_s^{in}$. If θ_s exceeds these limits, it is set equal to θ_s^{in} and $1.2\theta_s^{in}$, respectively, and θ_s is eliminated as a fitting parameter. The parameter δ was constrained by requiring $|\delta| > 0.0001$. Note that an investigation of the initial estimates and constraints was beyond the scope of the research, and it is likely that the parameter optimizations may be improved by selecting different initial estimates or constraints.

Results

Table 6 contains the number of data sets (n) that could be optimized for each combination of retention and conductivity functions and the corresponding median values for r^2 and MSE. The retention functions are identified in the first column and the numbers for the conductivity function are again given in the top row. Also included is the arithmetic mean (μ) of n, r^2, and MSE for individual retention functions, obtained as an average of the results for a particular $S_e(h)$ function for all K(h) and $K(S_e)$ functions, as shown in the last column of Table 6. Conversey, μ_n, μ_{r2}, and μ_{MSE} values for individual conductivity functions were obtained by averaging over results for all $S_e(h)$ functions.

The information in Table 6 was used to rank the performance of the hydraulic functions with respect to n, r^2, and MSE. Figure 2 shows the ranking for the $S_e(h)$ functions. On average,

Table 4. Parameter Combinations for Hydraulic Functions[a]

	K(h)				$K(S_e)$						
$S_e(h)$	31	32	33	34	41	42	43	44	45	46	47
11	α, β α, γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
12	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	α, β, γ $\alpha, \beta, \gamma, \delta$	α, β, γ δ	$\alpha, \beta, \gamma = 0$ γ, δ	$\alpha, \beta = \delta, \gamma = 0$ $\beta, \gamma, \delta = 1/2$	$\alpha, \beta, \gamma = 0$ γ, δ	$\alpha, \beta, \gamma = 0$ γ, δ	$\alpha, \beta, \gamma = 0$ γ, δ	α, β, γ γ, δ
13	α, β α, γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
14	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	α, β, γ $\alpha, \beta, \gamma, \delta$	α, β, γ δ	α, β, γ γ, δ	α, β, γ β, γ, δ	$\alpha, \beta, \gamma = 1$ γ, δ	$\alpha, \beta, \gamma = 1$ α, γ, δ	$\alpha, \beta, \gamma = 1$ α, γ, δ	$\alpha, \beta, \gamma = 1$ α, γ, δ
15	α, β γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	α, β β, γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
16	α, β α, γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
17	α, β α, γ, δ	α, β δ	α, β α, δ	—	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
18	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	α, β, γ $\alpha, \beta, \gamma, \delta$	α, β, γ δ	$\alpha, \beta, \gamma = 1$ γ, δ	$\alpha, \beta = \delta, \gamma = 1$ $\beta, \gamma, \delta = 1/2$	$\alpha, \beta, \gamma = 1$ γ, δ	$\alpha, \beta, \gamma = 1$ γ, δ	$\alpha, \beta, \gamma = 1$ γ, δ	$\alpha, \beta, \gamma = 1$ γ, δ
19	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	α, β, γ $\alpha, \beta, \gamma, \delta$	α, β, γ δ	$\alpha, \beta, \gamma = 2$ γ, δ	$\alpha, \beta = \delta, \gamma = 2$ $\beta, \gamma, \delta = 1/2$	$\alpha, \beta, \gamma = 2$ γ, δ	$\alpha, \beta, \gamma = 2$ γ, δ	$\alpha, \beta, \gamma = 2$ γ, δ	$\alpha, \beta, \gamma = 2$ γ, δ
20	α, β α, γ, δ	α, β δ	α, β α, δ	—	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
21	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	α, β, γ $\alpha, \beta, \gamma, \delta$	α, β, γ δ	$\alpha, \beta, \gamma = 1/2$ γ, δ	$\alpha, \beta = \delta, \gamma = 1/2$ $\beta, \gamma, \delta = 1/2$	$\alpha, \beta, \gamma = 1/2$ γ, δ	$\alpha, \beta, \gamma = 1/2$ γ, δ	$\alpha, \beta, \gamma = 1/2$ γ, δ	$\alpha, \beta, \gamma = 1/2$ γ, δ
22	α, β, γ $\alpha, \gamma = 1, \delta$	α, β, γ δ	α, β, γ α, δ	β, γ β, γ, δ	α, β, γ δ	$\alpha, \beta, \gamma = 3/4$ γ, δ	$\alpha, \beta = \delta, \gamma = 3/4$ $\beta, \gamma, \delta = 1/2$	$\alpha, \beta, \gamma = 3/4$ γ, δ	$\alpha, \beta, \gamma = 3/4$ γ, δ	$\alpha, \beta, \gamma = 3/4$ γ, δ	$\alpha, \beta, \gamma = 3/4$ γ, δ
23	α, β α, γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ
24	α, β α, γ, δ	α, β δ	α, β α, δ	α, β α, β, δ	α, β δ	α, β γ, δ	$\alpha, \beta = \delta$ $\beta, \gamma, \delta = 1/2$	α, β γ, δ	α, β γ, δ	α, β γ, δ	α, β γ, δ

[a] Retention and conductivity parameters are given in the top and bottom line of each section.

function 17 described the most data sets (417) and received a rank of 1; it was followed by functions 14 (412), 16 (410), and 11 (409). The last column of Table 6 shows that function 22 only fitted an average of 109 functions; of course, the number of fitted data sets may be increased by modifying the optimization algorithm or initial estimates. We were able to improve the results for combination 2234 by changing the initial estimate for γ. The rankings for the $K(S_e)$ and K(h) functions are given in Figures 3 and 4, respectively. The most data sets could be described with functions 34 and 47, respectively.

Table 5. Initial Estimates for Hydraulic Parameters

Texture[a]	θ_r	θ_s	α *l/cm*	β	γ[b]	δ	K_s *cm/d*
Sand	0.045	0.43	0.145	2.68	0.62	0.5	713
Loamy sand	0.057	0.41	0.124	2.28	0.56	0.5	350
Sandy loam	0.065	0.41	0.075	1.89	0.47	0.5	106
Loam	0.078	0.43	0.036	1.56	0.36	0.5	25.0
Silt	0.034	0.46	0.016	1.37	0.27	0.5	6.00
Silt loam	0.067	0.45	0.020	1.41	0.29	0.5	10.8
Sandy clay loam	0.100	0.39	0.059	1.48	0.32	0.5	31.4
Clay loam	0.095	0.41	0.019	1.31	0.24	0.5	6.24
Silty clay loam	0.089	0.43	0.010	1.23	0.19	0.5	1.68
Sandy clay	0.100	0.38	0.027	1.23	0.19	0.5	2.88
Silty clay	0.070	0.36	0.001	1.09	0.08	0.5	0.48
Clay	0.068	0.38	0.001	1.09	0.08	0.5	4.80

[a] According to USDA classification scheme.
[b] Initial estimate is 2.0 for combination 2234.

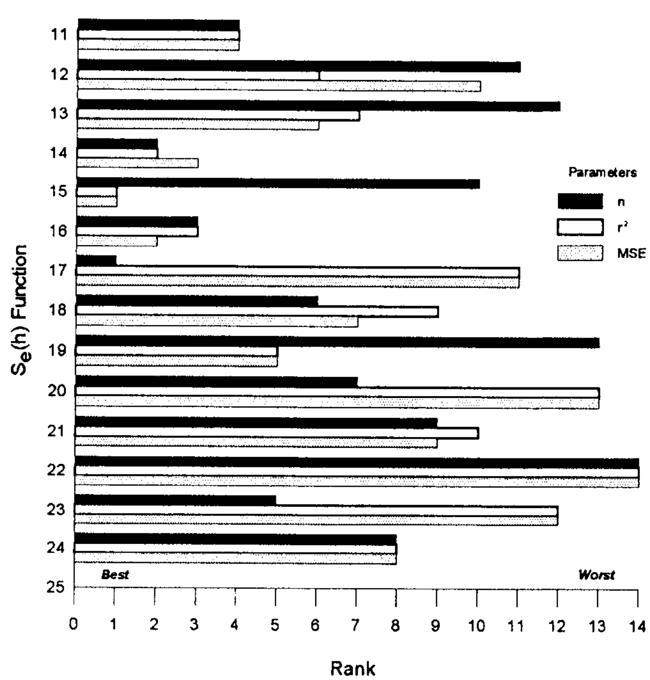

Fig. 2. Relative ranking of $S_e(h)$ functions.

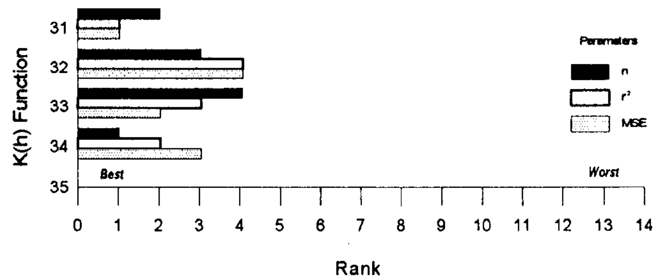

Fig. 3. Relative ranking of K(h) functions.

The rankings for r^2 and MSE are closely related. Function 15 has the highest r^2 (.910) and lowest MSE (.947) for fitting $S_e(h)$ data with function 14 having the next best r^2 and function 16 the next best MSE. Both functions 15 and 16 use only four parameters (Table 1). The MSE for function 16 (1.07) is considerably greater than for function 15 but it is obtained for a somewhat larger data set (410) than for function 15 (375). Figure 3 illustrates that the fitting of K(h) data is best done with function 31 in view of its high rankings for r^2 and MSE. The relatively poor performance of function 32 was already alluded to by Gardner (1958) — this expression may still be useful for conductivities close to saturation. Functions 33 and 34 appear to provide a reasonable description of the data. In view of the additional effort associated in using model 34 (cf. Table 3), we did not further pursue its use or the development of conductivity functions based on other models than Mualem's equation (2). According to Figure 4, functions 41 and 43 appear to provide the best description of $K(S_e)$ data. The reader may study Table 6 to make additional inferences regarding the suitability of particular functions or combinations of conductivity and retention functions.

Table 6. Number of Fitted Data Sets (n) and Median Value for r^2 and MSE[a]

		K(h)				$K(S_e)$							
S_e		31	32	33	34	41	42	43	44	45	46	47	μ
11	n	320	314	320	318	454	466	466	422	455	483	484	409
	r^2	.956	.858	.946	.945	.887	.888	.891	.901	.875	.883	.852	.898
	MSE	.673	1.85	.648	.784	1.05	1.15	1.13	.923	1.25	1.17	1.67	1.12
12	n	217	213	209	299	366	461	458	360	450	461	464	360
	r^2	.958	.870	.953	.947	.911	.854	.861	.868	.846	.863	.830	.887
	MSE	.704	2.05	.765	.865	1.01	1.85	1.87	1.66	2.01	1.70	2.19	1.52
13	n	276	200	65	321	343	345	375	302	357	307	456	304
	r^2	.893	.864	.821	.887	.899	.912	.891	.891	.892	.909	.847	.882
	MSE	1.23	2.06	2.31	1.39	1.04	.921	1.07	1.18	1.23	.969	1.71	1.37
14	n	321	316	321	321	466	486	483	385	465	486	487	412
	r^2	.967	.861	.963	.912	.905	.901	.892	.906	.879	.890	.864	.904
	MSE	.524	1.93	.559	1.17	1.10	.955	1.07	.840	1.25	1.20	1.63	1.11
15	n	320	222	320	320	431	379	482	358	410	465	422	375
	r^2	.964	.867	.958	.945	.907	.906	.898	.915	.891	.890	.871	.910
	MSE	.566	1.35	.537	.632	.893	.807	1.05	.776	1.16	1.05	1.60	.947
16	n	320	315	321	321	466	483	473	374	468	487	487	410
	r^2	.961	.858	.951	.944	.894	.894	.899	.906	.882	.885	.860	.903
	MSE	.606	1.81	.583	.649	.996	1.04	1.11	.940	1.22	1.16	1.68	1.07
17	n	305	313	308	—	452	475	468	419	460	482	485	417
	r^2	.880	.803	.874	—	.869	.861	.866	.865	.846	.853	.823	.854
	MSE	1.47	2.31	1.46	—	1.33	1.44	1.36	1.33	1.48	1.40	2.14	1.57
18	n	318	310	316	177	437	486	475	463	471	486	487	402
	r^2	.958	.856	.956	.852	.904	.884	.875	.888	.859	.860	.739	.875
	MSE	.587	2.06	.635	1.67	1.05	1.22	1.37	1.14	1.37	1.34	2.91	1.40
19	n	308	274	301	303	358	131	397	115	162	160	162	243
	r^2	.967	.848	.958	.947	.862	.914	.865	.923	.890	.875	.719	.888
	MSE	.520	2.08	.565	.827	1.38	.858	1.27	.753	1.03	1.31	2.36	1.18
20	n	259	314	132	—	453	484	469	443	450	338	464	381
	r^2	.925	.838	.566	—	.886	.888	.888	.887	.872	.820	.815	.839
	MSE	1.03	1.92	4.95	—	1.18	1.25	1.31	1.31	1.33	1.88	2.22	1.84
21	n	246	257	259	226	369	486	469	386	460	485	487	375
	r^2	.968	.869	.966	.936	.911	.829	.844	.866	.816	.822	.796	.875
	MSE	.457	1.95	.553	.938	.846	1.85	1.84	1.54	2.07	1.99	2.45	1.50
22	n	66	58	60	320	145	83	68	106	88	88	115	109
	r^2	.908	.819	.886	.932	.865	.472	.908	−2.82	.376	.701	.090	.376
	MSE	.970	2.19	1.12	.915	1.61	4.59	.699	41.9	9.13	4.03	9.16	6.94
23	n	318	315	319	240	456	484	426	459	469	479	485	405
	r^2	.886	.807	.882	.777	.867	.871	.879	.874	.849	.841	.792	.848
	MSE	1.38	2.30	1.36	2.44	1.23	1.33	1.24	1.29	1.38	1.54	2.58	1.64
24	n	244	287	236	305	442	447	431	391	438	441	476	376
	r^2	.947	.842	.927	.748	.895	.899	.905	.904	.864	.879	.839	.877
	MSE	.886	2.21	1.14	2.63	1.15	1.02	1.06	1.04	1.46	1.28	2.01	1.45
	μ_n	274	265	249	289	403	407	424	356	400	403	426	
	μ_{r2}	.938	.847	.901	.898	.890	.855	.883	.627	.831	.855	.767	
	μ_{MSE}	.828	2.01	1.23	1.25	1.13	1.45	1.25	4.05	1.96	1.57	2.59	

[a] MSE $\times 10^2$.

A more detailed comparison was subsequently made for the results of the most promising functions to a uniform data set. First we determined six $S_e(h)$, two $K(h)$, and four $K(S_e)$ functions with the best combined ranking for r^2 and MSE. Secondly, retention functions 11, 13, 14, 15, 16, and 19 were combined with conductivity functions 31, 33, 41, 42, 43, and 46 to select combinations that tended to have the best combined ranking for n, r^2, and MSE in Table 6. We used the seven combinations 1431, 1433, 1531, 1533, 1631, 1633, and 1931 for $S_e(h)$-$K(h)$ data and eight combinations 1441, 1442, 1443, 1541, 1543, 1641, 1642, and 1643 for $S_e(h)$-$K(S_e)$ data. It should be noted that combinations 1342 and 1542 could describe relatively few data sets and they were omitted despite their favorable combined ranking relative to some of the selected combinations. Thirdly, 306 $S_e(h)$-$K(h)$ data sets were identified that could be described by each of the above seven combinations, and, similarly, 401 $S_e(h)$-$K(S_e)$ data sets were found for which a parameter set could be optimized for all eight combinations of $S_e(h)$ and $K(S_e)$ functions.

Table 7 shows the extended results for the $S_e(h)$-$K(h)$ data. All combinations appear to describe the hydraulic data reasonably well with similar values for r^2 and MSE. Combinations 1931 and 1431 have the lowest median and mean value for MSE, respectively. Now that a comparison for the same data sets is being made, retention functions 15 and 16 appear somewhat less attractive than depicted in Figure 2. It is apparent that 31 is the preferred $K(h)$ function. It is less clear what function is best used to describe corresponding retention data. Based on the mean values for r^2 and MSE, the order of preference may be functions 14, 15 or 19, and 16. Further notice in Table 7 the similar median values for θ_r and θ_s, which reflect the constraints that were imposed on these parameters during the optimization. There is a big difference between the median and mean for α and especially

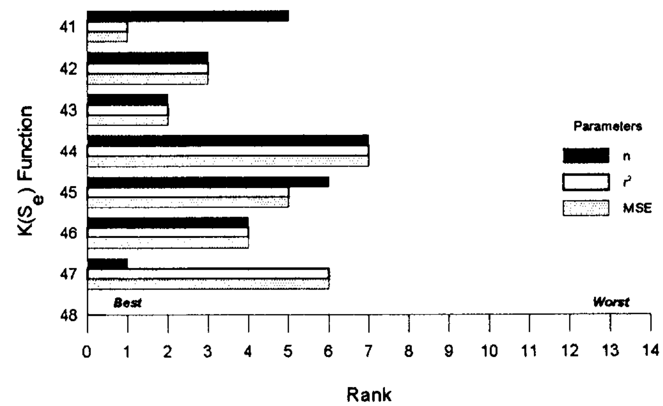

Fig. 4. Relative ranking of $K(S_e)$ functions.

for K_s while SE is relatively large. This suggests a skewed probability density function of these parameters. Clearly, K_s is not an attractive fitting parameter and it would be better to use a value for the conductivity at a higher suction or omit K_s as a fitting parameter altogether. The remaining parameters, particularly β and γ, appear to have a more favorable SE. Note that γ does not occur in combinations 1533 and 1633 (cf. Table 4).

Values for r^2, MSE, and parameters for fitting $S_e(h)$-$K(S_e)$ data with eight combinations are given in Table 8. A comparison of Tables 7 and 8 reveals that the data are not as well described as the $S_e(h)$-$K(h)$ data. The considerable number of outlying results causes a discrepancy between the median and arithmetic mean. Examination of the mean for r^2 and MSE indicates that function 43 tends to give the poorest results among the three $K(S_e)$ functions. Notice that a better performance was obtained

Table 7. Median, Mean, and Standard Error (SE) of r^2, MSE[a], and the Parameters of Seven Combinations Fitted to 306 $S_e(h)$-$K(h)$ Data Sets

$S_e(h)$-$K(h)$		1431	1433	1531	1533	1631	1633	1931
r^2	median	.965	.963	.962	.957	.958	.951	.966
	mean	.923	.911	.920	.905	.913	.897	.916
	SE	.0067	.0080	.0070	.0079	.0073	.0083	.0076
MSE	median	.530	.561	.566	.537	.618	.589	.520
	mean	1.31	1.71	1.49	1.49	1.53	1.53	1.47
	SE	.178	.333	.222	.236	.219	.223	.209
θ_r	median	.033	.057	.057	.057	.057	.057	.000
	mean	.028	.036	.051	.047	.053	.052	.009
	SE	.0016	.0015	.0001	.0012	.0007	.0009	.0012
θ_s	median	.465	.470	.460	.463	.461	.467	.461
	mean	.506	.506	.502	.504	.504	.506	.504
	SE	.0071	.0067	.0065	.0065	.0064	.0065	.0069
α	median	.059	.067	.067	.082	.097	.120	.111
1/cm	mean	.644	.289	.715	.487	.960	.492	12.5
	SE	.224	.117	.338	.156	.497	.144	5.52
β	median	1.08	1.09	1.21	1.19	.196	.185	11.1
	mean	1.98	1.83	1.53	1.46	.516	.440	104.2
	SE	.183	.154	.051	.046	.047	.037	32.1
γ	median	.207	.193	.653	—	.677	—	.257
	mean	.297	.279	1.11	—	1.07	—	33.6
	SE	.016	.015	.078	—	.073	—	7.24
δ	median	2.72	.155	4.30	.135	3.90	.113	2.64
	mean	3.95	.432	6.17	.418	5.79	.392	3.44
	SE	.294	.057	.348	.058	.400	.070	.149
K_s	median	18.9	14.7	18.3	18.0	28.6	26.0	33.6
cm/d	mean	7041	5008	4028	4168	7784	7872	14648
	SE	3925	2806	1308	1307	3528	3523	4150

[a] MSE × 10^2.

Table 8. Median, Mean, and Standard Error (SE) of r^2, MSE[a], and the Parameters of Eight Combinations Fitted to 401 $S_e(h)$-$K(S_e)$ Data Sets

$S_e(h)$-$K(S_e)$		1441	1442	1443	1541	1543	1641	1642	1643
r^2	median	.910	.913	.904	.909	.912	.907	.913	.913
	mean	.846	.847	.770	.844	.745	.844	.846	.782
	SE	.0090	.0090	.0675	.0091	.0756	.0090	.0090	.0673
MSE	median	.858	.847	.898	.868	.849	.808	.856	.853
	mean	3.23	3.19	4.06	2.73	4.85	2.78	3.12	4.33
	SE	.400	.444	1.01	.290	1.35	.300	.432	1.15
θ_r	median	.057	.034	.057	.057	.057	.057	.057	.057
	mean	.042	.029	.038	.045	.047	.046	.038	.041
	SE	.0012	.0014	.0013	.0011	.0010	.0010	.0013	.0012
θ_s	median	.410	.410	.410	.410	.410	.410	.410	.410
	mean	.451	.449	.447	.451	.447	.453	.448	.452
	SE	.0035	.0035	.0034	.0036	.0034	.0034	.0035	.0036
α	median	.033	.055	.050	.072	.061	.114	.101	.102
1/cm	mean	8.76	15.5	.689	3.88	3.34	4.62	4.16	5.35
	SE	7.42	10.8	.280	2.09	1.70	2.47	2.13	3.30
β	median	1.14	1.36	1.48	1.31	1.33	.264	.247	1.99
	mean	1.96	2.63	2.57	1.66	1.68	.604	.554	4.18
	SE	.183	.154	.051	.046	.047	.037	32.1	.283
γ	median	.339	.196	.201	—	.149	—	.129	.116
	mean	.673	.256	.447	—	.440	—	.150	.601
	SE	.049	.011	.084	—	.119	—	.004	.146
δ	median	.363	0.00	0.00	.370	0.00	.375	−3.60	.249
	mean	.997	−.72	1.50	2.60	−1.33	7.88	−5.44	.530
	SE	.203	.536	.247	1.55	.251	5.66	.525	.035
K_s	median	69.6	673	416	68.9	520	69.1	1406	304
cm/d	mean	12757	32321	27275	9905	27363	9006	58491	14189
	SE	3235	4584	5282	2731	4837	2449	6651	3495

[a] $MSE \times 10^2$.

for function 43 than 42 using the variably sized data set (Table 6). If we compare the means of r^2 and MSE for combinations 1441, 1541, and 1641 it appears that retention functions 15 and 16 perform slightly better than 14. The results reported in Table 7 also indicate that these three functions are attractive for describing retention data. The behavior of the functional parameters in Table 8 tends to be similar to that in Table 7. However, the parameter δ, which is now exclusively a power of the saturation except for combination 1643, exhibits a greater standard error. The importance of this parameter in the model of Mualem (1976a) has been investigated by a number of authors (Schuh and Cline, 1990; Yates et al., 1992). Yates et al. (1992) concluded that this parameter may fluctuate considerably while an independent means of predicting its value will not necessarily lead to a better prediction of the hydraulic conductivity. Hence we are not as concerned about the effect of δ on the accuracy of the optimization compared to K_s.

Conclusions

We attempted to fit the 14 retention and 11 conductivity functions that are shown in Tables 1 through 3 to 346 $S_e(h)$-$K(h)$ and 557 $S_e(h)$-$K(S_e)$ data sets of the unsaturated hydraulic database UNSODA. Table 6 presents the number of data sets that could be described with each of the 152 combinations of retention and conductivity functions as well as the corresponding median values for r^2 and MSE. Subsequently, we screened the results for desirable combinations of retention and conductivity functions (this amounted to the omission of a few attractive combinations such as 1334 and 2234). Seven combinations of $S_e(h)$ and $K(h)$ functions with favorable optimization results were fitted to 306 data sets (cf. Table 7). The best mean values for r^2 and MSE were obtained with retention functions 14 and 15 (van Genuchten, 1980), 19 (Globus, 1987), and 16 (Hutson and Cass, 1987). Function 31 of Gardner (1958) was the most attractive for describing $K(h)$ data. Eight combinations were used for fitting 401 $S_e(h)$-$K(S_e)$ data sets (cf. Table 8). Retention functions 14, 15, and 16 performed reasonably well, although function 14 has a slightly higher MSE than the other two functions when used with conductivity function 41. Examination of the mean for r^2 and MSE indicates that functions 41 (Brooks and Corey, 1964) and 42 (van Genuchten, 1980) can fit $K(S_e)$ data with comparable accuracy. In view of the relatively poor performance of the $K(S_e)$ functions, the use and development of alternative $K(S_e)$ functions seems appropriate.

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