An Analytical Solution to Saturated Flow in a Finite Stratified Aquifer[a]

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ABSTRACT

An analytical solution for the flow of water in a saturated-stratified aquitard-aquifer-aquitard system of finite length is presented. The analytical solution assumes one-dimensional horizontal flow in the aquifer and two-dimensional flow in the aquitards. Several examples are given which describe the use of the analytical solution. The horizontal flow assumption in the aquifer appears to be approximately valid when the hydraulic conductivity of the aquitards is less than or equal to about 10 percent of the aquifer value. A comparison is made between the analytical solution and a saturated-unsaturated finite-element solution for a situation where the upper layer is both saturated and unsaturated. For the situation investigated, the comparison indicates that the analytical solution provides an alternative to numerical models even when the upper layer is partially saturated.

Key words: saturated flow, stratified aquifer, analytical solution.

INTRODUCTION

In a recent paper, Beck *et al.* (1987, in review) described some of the considerations necessary when constructing a macroscale laboratory stratified aquifer-aquitard system. Part of the analysis consisted of determining the flow pattern in the aquifer. Since the overall objective was to maximize the distance over which the flow in the aquifer is approximately constant and horizontal (i.e., one-dimensional), a series of numerical simulations were carried out to investigate various length-width ratios and inlet-outlet sizes. These numerical simulations were carried out using the finite-element method and were quite costly and time-consuming. Of equal importance were the costs associated with the design and implementation of the finite-element grid system. For each length to width ratio considered, a new grid system was needed. To reduce costs, a preprocessor was developed to do many of the routine calculations. Although this reduced the man-hours required to set up the grid system, the costs of developing and using the preprocessor for each change in aquifer configuration were considerable.

Analytical solutions offer one means for reducing the computational costs compared to numerical techniques (Javandel *et al.*, 1984). This is especially true when the physical boundaries are simple and/or the ultimate goal is to obtain a steady-state solution of some physical property, since many numerical modeling codes must march through time towards the steady-state solution as opposed to calculating it directly (which some codes will do).

The purpose of this paper is to develop an analytical solution for the flow of water in a saturated-stratified aquitard-aquifer system of finite length and consisting of three layers. The upper and lower layers are assumed to be aquitards or aquicludes and the middle layer an aquifer. Although the hydraulic properties between layers may be different, it is assumed that within a layer the hydraulic properties are homogeneous and isotropic. A solution for an anisotropic system can be found, in general (see Selim, 1987, for an anisotropic solution of a problem with a different geometry), but is not developed here since isotropic media will be produced during construction of a physical stratified aquifer system. After solving the system of equations, the solution will be illustrated by examples. The first series of examples is for an hypothetical stratified aquifer 100 m in length. The final example compares the results from the analytical solution with those from Beck *et al.* (1987) who simulated partially saturated flow in the upper layer using a saturated-unsaturated finite-element solution developed by Wagner *et al.* (1985).

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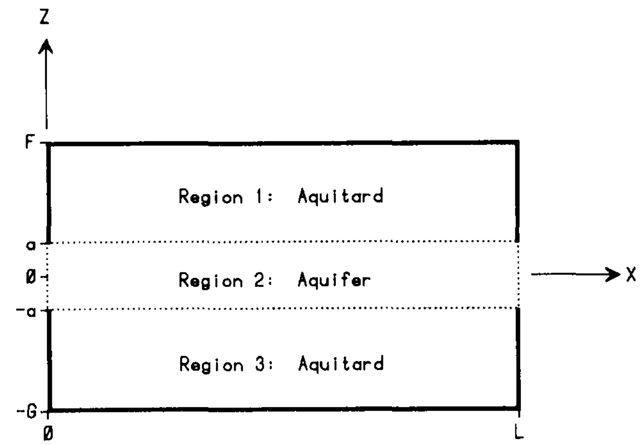

Fig. 1. Schematic diagram of the stratified aquifer-aquitard system where the bold solid and dotted lines indicate no-flow and prescribed head boundaries, respectively.

ANALYTICAL SOLUTION

The problem domain for which an analytical solution is sought is shown diagrammatically in Figure 1. In each of the three regions, the medium is assumed to be homogeneous and isotropic. This latter assumption is not required in general and the effects of anisotropic media can be included easily (see Selim, 1987). However, since the macroscopic laboratory aquifer (Beck *et al.,* 1987, in review) for which this solution was developed is assumed to be homogeneous and isotropic, the additional complexity of developing an anisotropic solution was deemed unnecessary. In the uppermost and lowermost layers (Regions 1 and 2), the flow of water is assumed to be two-dimensional whereas in the middle aquifer the flow of water is assumed to be one-dimensional. As a consequence of this latter assumption, the analytical solution described herein is strictly valid only as the ratio of the hydraulic conductivities (i.e., K_1/K_2 and K_3/K_2) approaches zero. The effects on the solution when these ratios are not zero will be shown in the examples section. Generally, accurate results occur for conductivity ratios up to and including about $K_1/K_2 = K_3/K_2 = 0.1$.

The equations which describe the flow of water in each region of Figure 1 subject to the listed assumptions are

$$\frac{\partial^2 H_1}{\partial x^2} + \frac{\partial^2 H_1}{\partial z^2} = 0 \qquad \text{Region 1} \qquad (1a)$$

$$K_1 \frac{d^2 H_2}{dx^2} - \frac{q_1(x)}{2a} + \frac{q_2(x)}{2a} = 0 \qquad \text{Region 2} \qquad (1b)$$

and

$$\frac{\partial^2 H_3}{\partial x^2} + \frac{\partial^2 H_3}{\partial z^2} = 0 \qquad \text{Region 3} \qquad (1c)$$

where H_i is the total hydraulic head for the i^{th} layer (i.e., $H_i = h_i + z$, where h_i is the pressure head); z is the vertical distance from an arbitrary reference level; and q(x) [L/T] represents the loss (for $0 \leqslant x < L/2$) or gain (for $L/2 < x \leqslant L$) of fluid from (or to) the aquifer and is defined to be positive in the positive x and z directions.

The boundary conditions considered for each region in the aquifer-aquitard system shown in Figure 1 are

Region 1:

$$\frac{\partial H_1}{\partial x} = 0 \quad \text{for } x = 0; \text{ and } x = L \quad \text{and } a \leqslant z \leqslant F$$
$$\frac{\partial H_1}{\partial z} = 0 \quad \text{for } 0 \leqslant x \leqslant L \quad \text{and } z = F \qquad (2a)$$

Region 2:

$$H_2 = H_0 \quad \text{for } x = 0; \quad \text{and } -a \leqslant z \leqslant a$$
$$H_2 = H_L \quad \text{for } x = L; \quad \text{and } -a \leqslant z \leqslant a \qquad (2b)$$

Region 3:

$$\frac{\partial H_3}{\partial x} = 0 \quad \text{for } x = 0; \text{ and } x = L \quad \text{and } -G \leqslant z \leqslant -a$$
$$\frac{\partial H_3}{\partial z} = 0 \quad \text{for } 0 \leqslant x \leqslant L \quad \text{and } z = -G \qquad (2c)$$

At the boundaries between regions, continuity requires the following conditions:

Boundary Between Regions 1 and 2:

for $0 \leqslant x \leqslant L$ and $z = a$

$$q_1(x) = -K_1 \frac{\partial H_1}{\partial z} = -K_2 \frac{\partial H_2}{\partial z} \quad \text{and } H_1 = H_2 \qquad (2d)$$

Boundary Between Regions 2 and 3:

for $0 \leqslant x \leqslant L$ and $z = -a$

$$q_2(x) = -K_2 \frac{\partial H_2}{\partial z} = -K_3 \frac{\partial H_3}{\partial z} \quad \text{and } H_2 = H_3 \qquad (2e)$$

The separation-of-variables technique (Churchill and Brown, 1978; Haberman, 1983; Wylie and Barrett, 1982) was used to solve equations (2a) and (2c), which, when written in terms of a generalized Fourier series, are

$$H_1(x,z) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(kx)\cosh[k(F - z)] \qquad (3a)$$

$$H_3(x,z) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos(kx)\cosh[k(G + z)] \qquad (3b)$$

where $k = n\pi/L$ are the eigenvalues, and $\cos(kx)$ the eigenfunctions. The constants A_n and B_n are found by using orthogonality.

Determining $q_1(x)$ and $q_2(x)$ by taking the derivative with respect to z in equations (3a) and (3b), evaluating the resulting expressions at $z = a$ and $-a$, respectively, substituting the results into equation (1b), and integrating with respect to x twice gives an expression for the hydraulic head in the aquifer

$$H_2(x) = p_1 \sum_{n=1}^{\infty} A_n \cos(kx)\sinh[k(F-a)]/k + p_2 \sum_{n=1}^{\infty} B_n \cos(kx)\sinh[k(G-a)]/k + Cx + D \quad (4)$$

where $p_1 = K_1/2aK_2$ and $p_2 = K_3/2aK_2$.

To complete the derivation, the constants of integration, C and D, and the Fourier coefficients, A_n and B_n, must be determined. To do this, the continuity equations at each of the aquifer-aquitard boundaries are used [equations (2d) and (2e)]. Equating the hydraulic heads at each of the boundaries provides a means for finding the Fourier coefficients. For the boundary between Region 1 and 2, this gives

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(kx)\{\cosh[k(F-a)] + \frac{p_1}{k}\sinh[k(F-a)]\} + p_2 \sum_{n=1}^{\infty} \frac{B_n}{k}\cos(kx)\sinh[k(G-a)] = Cx + D \quad (5a)$$

whereas for the boundary between Regions 2 and 3

$$\frac{B_0}{2} + p_1 \sum_{n=1}^{\infty} \frac{A_n}{k}\cos(kx)\sinh[k(F-a)] + \sum_{n=1}^{\infty} B_n \cos(kx)\{\cosh[k(G-a)] + \frac{p_2}{k}\sinh[k(G-a)]\} = Cx + D \quad (5b)$$

The eigenfunctions for this boundary value problem are orthogonal with respect to the weighting function, $\sigma(x) = 1$. Using the weighting function, equation (5) can be integrated to obtain Fourier coefficients in explicit form. Integrating and manipulating the expressions in equation (5) gives

$$A_0 = CL + 2D = B_0 \quad (6a)$$

$$A_n = \frac{4C/L}{k^2\cosh[k(F-a)]\{1+p_1\tanh[k(F-a)]+p_2\tanh[k(G-a)]\}} \quad \text{, n odd} \quad (6b)$$

$$= 0 \quad \text{, n even}$$

$$B_n = A_n \frac{\cosh[k(F-a)]}{\cosh[k(G-a)]} \quad (6c)$$

The final step necessary to complete the analytical solution is to determine the integration constants C and D. Substituting the values for the Fourier coefficients A_n and B_n from equation (6) into equation (4), rearranging and using the boundary conditions for H_2 at the inlet and outlet (i.e., at $x = 0$ and $x = L$) gives

$$H_2(x) = \frac{4C}{L} \sum_{\substack{n=1 \\ n,odd}}^{\infty} \zeta_n[\cos(kx) - 1] + Cx + H_0 \quad (7)$$

where

$$\zeta_n = \frac{p_1\tanh[k(F-a)] + p_2\tanh[k(G-a)]}{k^3\{1 + p_1\tanh[k(F-a)]/k + p_2\tanh[k(G-a)]/k\}} \quad \text{.......(8)}$$

$$C = \frac{(H_L - H_0)}{L - 8[\sum_{\substack{n=1 \\ n,odd}}^{\infty} \zeta_n]/L} \quad (9)$$

and, for completeness,

$$D = H_0 - \frac{4C}{L} \sum_{\substack{n=1 \\ n,odd}}^{\infty} \zeta_n \quad (10)$$

Equations (7) through (9) give the solution for the hydraulic head in the aquifer, and equations (3) and (6) give the solution for the head in the upper and lower layers.

Stream Lines

It is also of interest to be able to calculate the stream lines in the stratified aquifer system. Because of the horizontal flow assumption in Region 2 (i.e., the aquifer), the solution for the stream lines in this layer is trivial. However, since the flow in the upper and lower regions is two-dimensional, the stream function must be determined. The stream function can be calculated from the potential function (Kirkham and Powers, 1971) by using the Cauchy-Riemann relationships,

$$K\frac{\partial H}{\partial x} = \frac{\partial \psi}{\partial z} \quad \text{and} \quad -K\frac{\partial H}{\partial z} = \frac{\partial \psi}{\partial x} \qquad (11)$$

Integrating the left-hand part of equation (11) gives the stream function

$$\psi(x,y) = \int K\frac{\partial H}{\partial x}dz + F(x) = \int -q dz + F(x) \quad (12)$$

Doing likewise with the right-hand side of equation (11) gives an analogous equation in terms of the derivative of H with respect to z plus a function of integration of G(z). Equating the two relationships shows that the functions of integration are constants and without loss of generality can be taken to be zero.

To determine the stream functions for Regions 1 and 3, the derivatives of equation (3) with respect to x are incorporated into equation (12) and integrated, which gives

$$\psi_1(x,z) = K_1 \sum_{\substack{n=1\\ n,odd}}^{\infty} A_n \sin(kx)\sinh[k(F-z)] \quad (13a)$$

and

$$\psi_3(x,z) = K_3 \sum_{\substack{n=1\\ n,odd}}^{\infty} B_n \sin(kx)\sinh[k(G+z)] \quad (13b)$$

The maximum value for ψ_1 and ψ_3, respectively, occurs at the points (L/2, a) and (L/2, −a). Although the solution for the stream lines in the middle aquifer is trivial given the assumptions concerning the flow in the aquifer, the flux of water crossing a vertical plane as a function x can be found by taking the derivative of equation (7) with respect to x and incorporating it into the definition of the flux, and is

$$Q(x) = 2aCK_2[(4/L) \sum_{\substack{n=1\\ n,odd}}^{\infty} \zeta_n k \sin(kx) - 1] \quad (14)$$

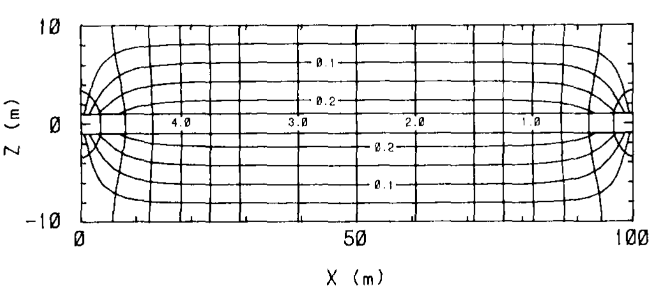

Fig. 2. Hydraulic head and stream functions when $K_1/K_2 = K_3/K_2 = 0.1$. The contour levels for the stream function are: 0.05, 0.1, 0.15, and 0.2. The contour levels for the head are given in the text.

The quantity $Q(0) = -2aCK_2 = Q(L)$ is the maximum flow in the middle layer (i.e., the aquifer) and can be used to normalize the stream function values [i.e., $\overline{\psi}(x,z) = \psi(x,z)/Q(0)$]. Subtracting the values from two normalized stream lines (i.e., $\overline{\psi}_a - \overline{\psi}_b$) gives the fraction of flow that passes through a plane that intersects the two normalized stream lines. Also, $\overline{\psi}(x,z)$ is defined such that the value of the stream function is zero at the no-flux boundaries.

EXAMPLES

The remaining part of this paper will illustrate the analytical solution for the flow in a saturated-stratified aquifer by example. In the following figures (with the exception of Figures 5 and 6) the potential and normalized-stream function lines were obtained by generating a matrix of values at n points using the solution contained herein. A contouring routine (Yates, 1987) was used to find the position of the contour lines which were then plotted. For Figures 2 through 4 and 7 through 8, respectively, 713 and 629 discrete values of hydraulic head and normalized stream function were used for contouring. The contour levels for hydraulic head in meters for Figures 2 through 4 are: 0.25, 0.5, 0.75, 1.0, 1.25, 1.5, 2.0, 2.5, 3.0, 3.5, 3.75, 4.0, 4.25, 4.5, and 4.75; and in Figures 7 and 8 are: 15.3, 15.35, 15.45, 15.55, 15.75, 16.05, 16.25, 16.75, 17.25, 17.45, 17.75, 17.95, 18.05, 18.15, and 18.2 cm. The contour levels for the stream function are given in each figure legend.

Shown in Figures 2 through 4 are contours of hydraulic head and stream function for a hypothetical stratified aquifer system 100 m in length and 20 m in depth. The aquifer is located in the interval $-1 < z < 1$ m (i.e., 2a = 2m) and the upper and lower aquitards are located in the intervals $1 < z < 10$ and $-10 < z < -1$ m, respectively. The hydraulic conductivity in the aquifer is 1 m/day and the values for the hydraulic conductivity ratios for the upper and lower aquitards (i.e., K_1/K_2 and K_3/K_2) varies in each of the figures.

In Figure 2 the hydraulic conductivity ratio is $K_1/K_2 = K_3/K_2 = 0.1$. The flow of water across the inlet (and outlet) boundary of the aquifer given by equation (14) is $Q(0) = Q(L) = 0.180$ m²/day. Dividing equation (13a) and (13b) by Q(0) gives the normalized stream function $\overline{\psi}(x,z)$ which has maximum values at $x = L/2$ and $z = \pm a$. In this form, the normalized stream function represents a flow fraction rather than the actual flow quantity. For the upper and lower aquitards, respectively, the maximum value for the normalized stream

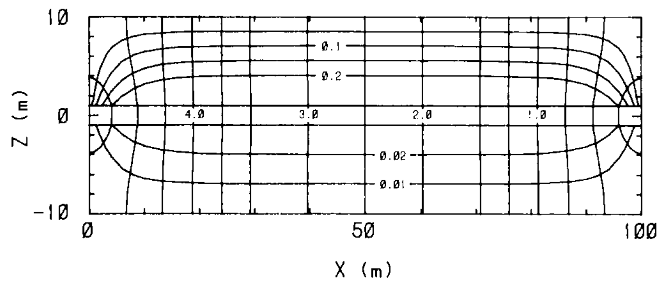

Fig. 3. Hydraulic head and stream functions when $K_1/K_2 = 0.1$ and $K_3/K_2 = 0.01$. The contour levels for the stream function are: 0.05, 0.01, 0.015, and 0.2 in the upper layer and are: 0.01 and 0.02 in the lower layer. The contour levels for the head are given in the text.

function is $\overline{\psi}_1 = \overline{\psi}_3 = 0.24$ which represents 47.4% of the flow into the stratified aquifer system. Therefore, between the points $x = 0$ and $x = L/2$, 47.4% of the flow leaves the aquifer and enters either the upper or lower region. In this example, the ratio of the hydraulic conductivity is about as large as it can be and still reasonably satisfy the horizontal flow assumption. Two methods for showing whether the horizontal flow assumption is approximately satisfied use the hydraulic head and flux profiles in the aquifer. This is discussed in greater detail in conjunction with Figures 5 and 6.

Figure 3 shows contours of the hydraulic head and normalized stream functions when the hydraulic conductivity ratios for the upper and lower layers are $K_1/K_2 = 0.1$ and $K_3/K_1 = 0.01$, respectively. The flux crossing the inlet and outlet boundary $Q(0) = Q(L)$, is 0.144 m^2/day and the maximum values for the normalized stream functions $\overline{\psi}_1$ and $\overline{\psi}_3$ are 0.30 and 0.03, respectively. Comparing Figures 2 and 3 shows that reducing one of the conductivity ratios reduces the overall flow in the upper and lower regions (i.e., 33.1% vs. 47.4%) but increases the percentage of flow in the aquitard with the larger conductivity ratio (i.e., 30.1% vs. 23.7%) and decreases the flow in the aquitard with the lower ratio (i.e., 3.0% vs. 23.7%) compared to the case shown in Figure 2.

Shown in Figure 4 is an example where the horizontal flow assumption is no longer valid. For this example, the hydraulic conductivities for each region were assumed to be equal and have a value of 1.0 m/day, giving conductivity ratios of 1.0. The flow across the aquifer inlet and outlet boundary is $Q(0) = Q(L) = 0.818$ m^2/day, and the maximum value for the normalized stream functions $\overline{\psi}_1$ and $\overline{\psi}_3$ is 0.450. For this example, 90% of the flow leaves the aquifer between the inlet and the point $x = L/2$. Although most of the flow passes through the upper and lower layers, there is still a large area in the middle of the aquifer where the flow is approximately horizontal. In particular, the flow in the aquifer where $5 \leqslant x \leqslant 95$ is approximately horizontal since only 10% of the total flow will pass out of the portion of the aquifer between $x = 5$ and $x = L/2$ and move through the upper and lower regions. This tends to support one's intuition that, in general terms, for a flow system with an inlet and outlet that is open over a small area at the ends of the aquifer, only the region near the inlet and outlet exhibits strongly two-dimensional behavior. If the flow pattern near the inlet and outlet is required, a solution which accounts for the two-dimensional flow in this area should be used. If, on the other hand, the region in the middle part of the aquifer is of interest, a solution which uses the horizontal flow assumption may, in general, produce adequate results. It should be kept in mind, however, that although the flow in the aquifer is predominantly horizontal, there is a relatively large change in flux with position near the inlet.

To demonstrate the validity of each of the examples shown in Figures 2 through 4, the hydraulic head and the flux as functions of position in the middle aquifer were plotted in Figures 5 and 6. For perfectly horizontal flow [i.e., $q_1(x) = q_2(x) = 0$] the solution for the hydraulic head in the middle aquifer is the straight line $H_1(x) = 5 - x/20$ and the gradient, and hence the flow rate, would be a constant throughout the aquifer. It is apparent from Figure 5 that the examples shown in Figures 2 and 3 (i.e., conductivity ratios of $K_1/K_2 = K_3/K_2 = 0.1$ and the ratios $K_1/K_2 = 0.1$ and $K_3/K_2 = 0.01$) give almost the same hydraulic head profile in the middle aquifer as the perfectly horizontal (one-dimensional) case. Comparison between the analytical solution con-

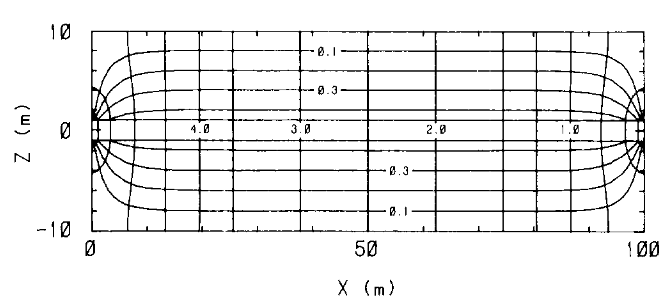

Fig. 4. Hydraulic head and stream functions when $K_1/K_2 = K_3/K_2 = 1.0$. The contour levels for the stream function are: 0.1, 0.2, 0.3, and 0.4. The contour levels for the head are given in the text.

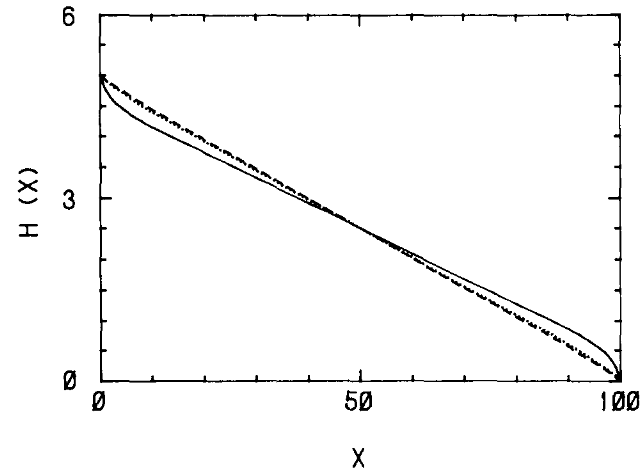

Fig. 5. Hydraulic head as a function of position in the aquifer. The dotted, dashed, and solid lines indicate the head profile for the examples shown in Figures 2, 3, and 4, respectively.

tained herein and the "straight-line" solution gives one means for determining the validity of the horizontal flow assumption. From this comparison, it appears that hydraulic conductivity ratios up to and including about 0.1 are consistent with the horizontal one-dimensional flow assumption. The third example contained in Figure 4, on the other hand, significantly violates the horizontal flow assumption. Only in the region $5 \leq x \leq 95$ m is the hydraulic head profile approximately linear (but with a different, i.e., lesser slope).

Shown in Figure 6 is the flow rate as a function of position through the aquifer. For each of the examples shown, there is a reduction in the flow rate in the aquifer near the inlet and an increase near the outlet point. Also, consistent

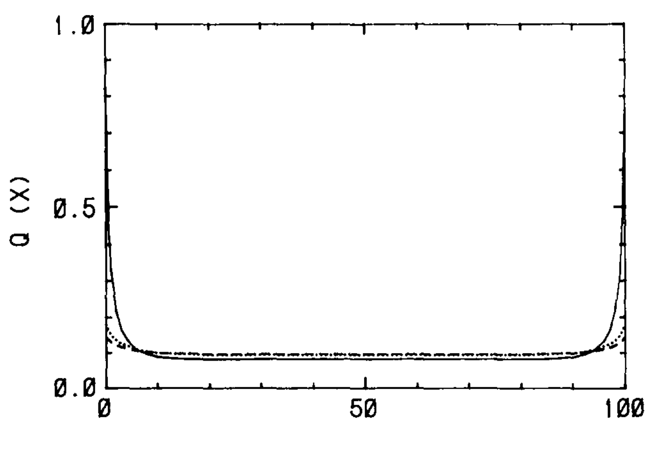

Fig. 6. Flux profile as a function of position in the aquifer. The dotted, dashed, and solid lines indicate the flux profile for the examples shown in Figures 2, 3, and 4, respectively.

with continuity requirements, the flow rate profile is symmetric about the point $x = L/2$. The flow rate profile also can be used to judge whether the model assumptions have been violated. For the examples shown in Figures 2 and 3, the reduction in the flow rate with position near the inlet is relatively small compared to Figure 4, again supporting the conclusion that a hydraulic conductivity ratio of 1.0 significantly violates the horizontal flow assumption. For the "straight-line" solution, the flow rate as a function of position would be $Q(x) = 2aK_2 \text{grad}(h) = 0.1\ m^2/\text{day}$ and is a constant.

As a final example, a comparison is made between the analytical solution contained herein and a saturated-unsaturated flow solution using the finite-element method for a stratified aquifer with a partially saturated upper region. The motivation for making this comparison is to see if the analytical solution will yield reasonable results when compared to the finite-element solution given the assumptions that the flow in the middle region is horizontal and that the unsaturated hydraulic conductivity in the upper aquifer can be taken as approximately equal to the saturated value. If the errors associated with using the analytical solution are small, then a savings in terms of computational costs and man-hours needed to set up the finite-element grid system would result by using the analytical solution instead of the finite-element solution.

Two macroscale laboratory aquifers, described in greater detail by Beck *et al.* (1987), are 488 cm in length, 122 cm in thickness, and 122 cm in width with a 61-cm surface layer of Lincoln fine sand, a 30.5-cm aquifer (middle layer) which consists of Owl Creek sand, and a 30.5-cm aquiclude (lower layer) consisting of Brick Plant soil. The saturated hydraulic conductivity values for the upper, middle, and lower layers are 0.2, 10.0, and .0001 cm/hr, respectively. The moisture retention data for the upper region which were used in the finite-element solution, were determined in the laboratory and fitted to the Clapp and Hornberger (1978) relationship using a nonlinear optimization technique. A summary of the soil coefficients is given in Table 1. The water table is located by the line: 18 cm-3 cm(x/L). The contour values for the hydraulic head from the finite-element solution and the analytical solution (assuming the conductivity in the upper region is a constant) are shown in Figure 7. With the exception of the corner areas in the upper layer, the results of the two methods are almost the same and demonstrate the utility

Table 1. Summary of Soil and Clapp and Hornberger* (1978) Coefficients for the Saturated and Unsaturated Upper Layer (Lincoln Fine Sand) in Figures 7 and 8

Coefficient	*Mean value*
θ_s	.455 (0.011)**
h_0	−41.65 (6.54)
b	1.71 (0.29)

* Clapp and Hornberger (1978) relationships:

$$K(\theta) = K_s(\theta/\theta_s)^{2b+3}$$

$$h(\theta) = h_0(\theta/\theta_s)^{-b}$$

(note: $H = h + z$)

where K_s, θ_s, h_0, and b are the saturated hydraulic conductivity, porosity, air entry pressure, and an empirical constant, respectively.

** Coefficient standard deviation resulting from nonlinear optimization technique.

of the analytical solution in describing the flow pattern in a stratified aquifer system even when it is partially unsaturated.

For a comparison between the two methods, values of the hydraulic head were calculated at the four points marked in Figure 7. At the points marked "1," "2," "3," and "4," respectively, the values of the hydraulic head which result from the analytical solution are: 18.05, 18.01, 17.95, and 17.93 cm whereas from the numerical solution are: 18.07, 18.04, 17.98, and 17.96 cm. This demonstrates that the rather large discrepancies between contour lines in the corners of the graph are due to small gradients in the hydraulic head.

The finite-element solution technique used here also was used in the design of the macroscale stratified aquifers by Beck *et al.* (1987) at a larger cost. In terms of design, an identical macroscale stratified aquifer system would have resulted if the slightly more restrictive (in terms of assumptions) analytical solution would have been available for use.

Another advantage the analytical solution has over many finite-element solutions is that the stream lines can be calculated easily. Shown in Figure 8 are the stream lines using equation (13) along with the hydraulic head values from the analytical solution. Figure 8 shows that 97% of the flow remains in the middle aquifer, and thus the horizontal flow assumptions are closely approximated.

The finite-element program used by Beck *et al.* (1987) does not provide the stream lines directly and would require using the matrix of head values and a finite-difference approach. Since large errors would be introduced by generating the stream lines in this manner, they are not included in Figure 8.

CONCLUSIONS

An analytical solution for the flow of water in a saturated-stratified aquitard-aquifer-aquitard system has been derived and demonstrated with examples. The major assumptions used in developing this analytical solution include: each layer is fully saturated and at steady-state, for each layer the medium is homogeneous and isotropic (although differences between layers are allowed), flow of water in the aquifer is horizontal and one-dimensional, and flow of water in the aquitards is two-dimensional. In the examples contained herein, it was found that the ratio of hydraulic conductivity between layers (i.e., K_1/K_2 and K_3/K_2) should be less than or equal to 0.1 for the horizontal flow assumption to be approximately valid. When the ratio of hydraulic conductivities is larger, a solution which allows two-dimensional flow in the aquifer should be used.

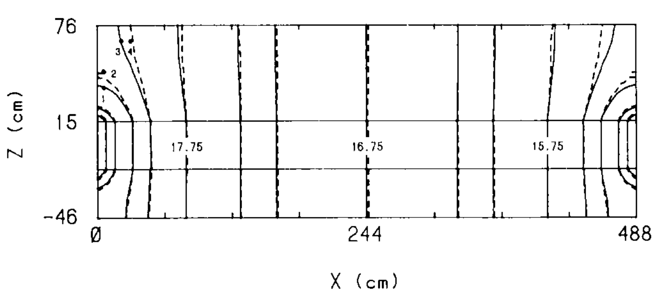

Fig. 7. Comparison between the analytical solution (solid line) and the finite-element solution (dashed line) when the upper layer is partially saturated. The water table is described by the line 18 cm-3 cm x/L. See the text for a discussion concerning the four points marked in this figure. The contour levels are given in the text.

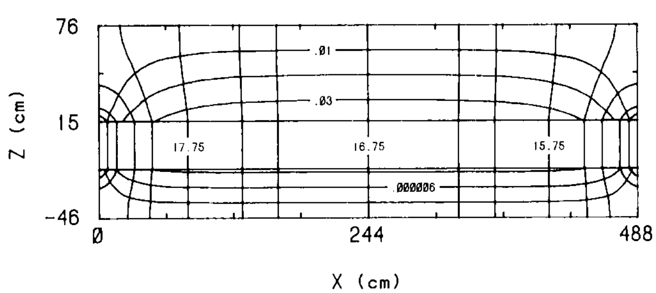

Fig. 8. Hydraulic head and stream function profiles from the analytical solution. The contour levels for the stream function are: 0.01, 0.02, and 0.03 in the upper layer and are: 3×10^{-6}, 6×10^{-6}, and 9×10^{-6} for the lower layer. The contour levels for the head are given in the text.

A comparison was also made between the results of the analytical solution and a finite-element solution when the upper layer was both saturated and unsaturated. The analytical solution along with the assumption that the unsaturated hydraulic conductivity was approximately equal to the saturated value was found to produce approximately the same hydraulic head profile as the finite-element solution with the exception of the corners of the aquifer. In terms of designing a macroscale stratified aquifer (see Beck *et al.*, 1987) identical results would have occurred regardless of whether the analytical or finite-element solution was used.

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